Math 3A03 Handout

The p-Norms on $\mathbb{R}^n$

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In class I mentioned that proving that the p-norms are actually norms on $\mathbb{R}^n$ is tricky because the proof depends on an inequality that might be unfamiliar. I asked ChatGPT to create a handout that gives a complete proof. After a little back and forth, it produced the following LaTeX document.

–David Earn

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad \text{for } 1 \leq p < \infty,$$

and define

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|.$$

We will prove that for every $p \in [1, \infty]$, the function $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.

Definition 1. A *norm* on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ such that for all $x, y \in \mathbb{R}^n$ and all $a \in \mathbb{R}$, the following properties hold:

1. *Positive definiteness:*

$$\|x\| \geq 0, \qquad \|x\| = 0 \iff x = 0.$$

2. *Absolute homogeneity:*

$$\|ax\| = |a| \; \|x\| .$$

3. *Triangle inequality:*

$$\|x + y\| \leq \|x\| + \|y\| .$$

The only substantial point for the p-norms is the triangle inequality when $1 < p < \infty$. For that we will use Hölder's inequality, which in turn follows from Young's inequality.

Lemma 1 (Young's inequality)**.** *Let* $1 < p < \infty$*, and let* q *be defined by*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then for all $u, v \geq 0$*,*

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Proof. Fix $v \geq 0$, and consider the function

$$f(u) := \frac{u^p}{p} - uv \qquad (u \geq 0).$$

Then

$$f'(u) = u^{p-1} - v.$$

Thus $f'(u) = 0$ when

$$u = v^{1/(p-1)}.$$

Since

$$f''(u) = (p-1)u^{p-2} \geq 0,$$

this critical point gives the minimum of f. Evaluating f there gives

$$f\left(v^{1/(p-1)}\right) = \frac{1}{p}\, v^{p/(p-1)} - v^{p/(p-1)} = -\left(1 - \frac{1}{p}\right) v^{p/(p-1)}.$$

Because $q = p/(p-1)$, this is

$$-\frac{v^q}{q}.$$

Therefore

$$f(u) \geq -\frac{v^q}{q},$$

that is,

$$\frac{u^p}{p} - uv \geq -\frac{v^q}{q}.$$

Rearranging yields

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

□

Lemma 2 (Hölder's inequality)**.** *Let* $1 < p < \infty$*, and let* q *be defined by*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then for all $x, y \in \mathbb{R}^n$*,*

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}.$$

Proof. If $x = 0$ or $y = 0$, then the inequality is immediate. So assume $x \neq 0$ and $y \neq 0$.

Set

$$A := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \qquad B := \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

Then $A > 0$ and $B > 0$. For each i, define

$$u_i := \frac{|x_i|}{A}, \qquad v_i := \frac{|y_i|}{B}.$$

Applying Lemma 1 to u_i and v_i, we obtain

$$u_i v_i \leq \frac{u_i^p}{p} + \frac{v_i^q}{q}.$$

Multiplying by AB, we get

$$|x_i y_i| \leq AB \left(\frac{u_i^p}{p} + \frac{v_i^q}{q} \right).$$

Summing over i yields

$$\sum_{i=1}^{n} |x_i y_i| \leq AB \left(\frac{1}{p} \sum_{i=1}^{n} u_i^p + \frac{1}{q} \sum_{i=1}^{n} v_i^q \right).$$

Now

$$\sum_{i=1}^{n} u_i^p = \sum_{i=1}^{n} \frac{|x_i|^p}{A^p} = \frac{\sum_{i=1}^{n} |x_i|^p}{A^p} = 1,$$

and similarly

$$\sum_{i=1}^{n} v_i^q = 1.$$

Hence

$$\sum_{i=1}^{n} |x_i y_i| \leq AB \left(\frac{1}{p} + \frac{1}{q} \right) = AB.$$

Substituting the definitions of A and B proves the result. □

Theorem 1. *For every $p \in [1, \infty]$, the function $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$.*

Proof. We treat separately the cases $1 \leq p < \infty$ and $p = \infty$.

Case 1: $1 \leq p < \infty$. We verify the properties in the definition of a norm.

Positive definiteness. Since each term $|x_i|^p$ is nonnegative,

$$\sum_{i=1}^{n} |x_i|^p \geq 0,$$

so $\|x\|_p \geq 0$.

Moreover,

$$\|x\|_p = 0 \iff \sum_{i=1}^{n} |x_i|^p = 0.$$

A sum of nonnegative numbers is zero if and only if every term is zero. Thus

$$|x_i|^p = 0 \quad \text{for every } i,$$

which is equivalent to $x_i = 0$ for every i. Hence

$$\|x\|_p = 0 \iff x = 0.$$

Absolute homogeneity. For $a \in \mathbb{R}$,

$$\|ax\|_p = \left(\sum_{i=1}^{n} |ax_i|^p\right)^{1/p} = \left(\sum_{i=1}^{n} |a|^p |x_i|^p\right)^{1/p} = |a| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = |a| \, \|x\|_p .$$

Triangle inequality when $p = 1$. In this case,

$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} \big(|x_i| + |y_i|\big) = \|x\|_1 + \|y\|_1 .$$

Triangle inequality when $1 < p < \infty$. Let q be the conjugate exponent, so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We begin with

$$\|x+y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_i| \, |x_i + y_i|^{p-1}.$$

Using $|x_i + y_i| \leq |x_i| + |y_i|$, we obtain

$$\|x+y\|_p^p \leq \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}.$$

Apply Hölder's inequality, Lemma 2, to the first sum:

$$\sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |x_i + y_i|^{(p-1)q}\right)^{1/q} .$$

Since

$$(p-1)q = (p-1)\frac{p}{p-1} = p,$$

this becomes

$$\sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} \leq \|x\|_p \, \|x+y\|_p^{p/q} .$$

But

$$\frac{p}{q} = p - 1,$$

so

$$\sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} \leq \|x\|_p \, \|x + y\|_p^{p-1}.$$

Similarly,

$$\sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1} \leq \|y\|_p \, \|x + y\|_p^{p-1}.$$

Therefore

$$\|x + y\|_p^p \leq \left(\|x\|_p + \|y\|_p\right) \|x + y\|_p^{p-1}.$$

If $\|x + y\|_p = 0$, then the desired inequality is immediate. Otherwise we divide by $\|x + y\|_p^{p-1}$ and obtain

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Thus $\|\cdot\|_p$ is a norm on $\mathbb{R}^n$ for every $1 \leq p < \infty$.

Case 2: $p = \infty$. Now let

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|.$$

Again we verify the properties in the definition of a norm.

Positive definiteness. Clearly $\|x\|_\infty \geq 0$. Also,

$$\|x\|_\infty = 0 \iff |x_i| = 0 \text{ for all } i \iff x = 0.$$

Absolute homogeneity. For $a \in \mathbb{R}$,

$$\|ax\|_\infty = \max_{1 \leq i \leq n} |ax_i| = |a| \max_{1 \leq i \leq n} |x_i| = |a| \, \|x\|_\infty.$$

Triangle inequality. For each i,

$$|x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|_\infty + \|y\|_\infty.$$

Taking the maximum over i gives

$$\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty.$$

Thus $\|\cdot\|_\infty$ is also a norm on $\mathbb{R}^n$. □

Remark. The restriction $p \geq 1$ is essential. For $0 < p < 1$, the formula

$$\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

does not define a norm, because the triangle inequality fails.