Math 3A03 Handout

Trigonometric Differentiability and Compactness

25 February 2026

NOTE: In Lecture 18 on 24 Feb 2026, we had a brief Q&A session at the end of the class. On 25 Feb 2026, I e-mailed the class with sketches of answers to questions I didn't answer or didn't fully answer. Later I gave the following prompt to ChatGPT 5.2:

> I sent the following e-mail to my real analysis class. Can you use it as a basis for a fully self-contained and very pedagogical latex handout? I've also attached my slides for the differentiation and integration sections.

Below is the LaTeX response from ChatGPT that I received 6 minutes later. I have not edited it. Note that the proof that $[0, 1]$ is compact is different from the proof I gave in my solutions to Assignment 3, so it is instructive to see that. As a matter of taste I might express some things in ChatGPT's text slightly differently, but I don't see any errors (please let me know if you do) and, quite honestly, there are some parts that are probably better presented than I would have managed had I spent the many hours it would have taken me to produce a 7 page document of this type myself. It is humbling.

– David Earn

Context

This handout expands on three questions that came up in a test-prep Q&A:

1. How do we prove cos is differentiable when cos is defined using an integral (sectoral area)?

2. Is the intersection of compact sets compact?

3. How does the Heine–Borel theorem get proved?

Everything is written to be *self-contained* (within standard results from the course: the Fundamental Theorem of Calculus, the Mean Value Theorem, and the Inverse Function Theorem).

1 Defining cos and sin from sectoral area

The key idea is that the angle θ (in radians) is characterized by a *geometric area*: the area of a sector of the unit disk is $\theta/2$. We build this into a definition by turning "sector area" into a function of an x–coordinate on the unit circle.

1.1 A formula for sectoral area

Consider the upper semicircle $y = \sqrt{1 - x^2}$ for $-1 \le x \le 1$. For each $x \in [-1, 1]$, define

$$\mathcal{A}(x) \;=\; \frac{x\sqrt{1 - x^2}}{2} \;+\; \int_x^1 \sqrt{1 - t^2}\, dt. \tag{1}$$

Geometrically, $\mathcal{A}(x)$ is the area of the sector swept out (counterclockwise) from $(1, 0)$ to the point $\left(x, \sqrt{1 - x^2}\right)$ on the upper semicircle; in particular, $\mathcal{A}(1) = 0$ and $\mathcal{A}(-1) = \pi/2$.

Remark. One can also define inverse trig functions directly by integrals such as

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt \qquad (-1 < x < 1),$$

and then define sin as the inverse of arcsin. The sectoral-area route we use here is closely related, but matches the approach taken in our integration slides.

1.2 $\mathcal{A}$ is differentiable and strictly decreasing

Lemma 1.1 (Differentiability and derivative of $\mathcal{A}$). *The function $\mathcal{A}$ is differentiable on $(-1, 1)$ and*

$$\mathcal{A}'(x) = -\frac{1}{2\sqrt{1 - x^2}} \quad \text{for } -1 < x < 1. \tag{2}$$

In particular, $\mathcal{A}'(x) < 0$ on $(-1, 1)$, so $\mathcal{A}$ is strictly decreasing on $(-1, 1)$.

Proof. Write $\mathcal{A}(x) = g(x) + h(x)$ with

$$g(x) = \frac{1}{2}x\sqrt{1 - x^2}, \qquad h(x) = \int_x^1 \sqrt{1 - t^2}\, dt.$$

For $-1 < x < 1$, g is differentiable by the product and chain rules. Differentiate:

$$g'(x) = \frac{1}{2}\left(\sqrt{1 - x^2} + x \cdot \frac{1}{2\sqrt{1 - x^2}}(-2x)\right) = \frac{1}{2}\left(\sqrt{1 - x^2} - \frac{x^2}{\sqrt{1 - x^2}}\right).$$

Combine terms:

$$g'(x) = \frac{1}{2} \cdot \frac{(1 - x^2) - x^2}{\sqrt{1 - x^2}} = \frac{1 - 2x^2}{2\sqrt{1 - x^2}}.$$

Next, the Fundamental Theorem of Calculus (differentiation under the integral sign for variable limits) gives

$$h'(x) = -\sqrt{1 - x^2}.$$

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Therefore

$$\mathcal{A}'(x) = g'(x) + h'(x) = \frac{1 - 2x^2}{2\sqrt{1 - x^2}} - \sqrt{1 - x^2} = \frac{1 - 2x^2 - 2(1 - x^2)}{2\sqrt{1 - x^2}} = -\frac{1}{2\sqrt{1 - x^2}},$$

which is (2). Since the derivative is strictly negative, $\mathcal{A}$ is strictly decreasing on $(-1, 1)$. $\square$

It is often convenient to remove the factor $\frac{1}{2}$:

Definition 1.1 (Twice the sectoral area)**.** Define $\mathcal{B} : [-1, 1] \to \mathbb{R}$ by

$$\mathcal{B}(x) = 2\mathcal{A}(x).$$

Then Lemma 1.1 implies $\mathcal{B}$ is differentiable on $(-1, 1)$ with

$$\mathcal{B}'(x) = -\frac{1}{\sqrt{1 - x^2}} \quad (-1 < x < 1), \tag{3}$$

so $\mathcal{B}$ is strictly decreasing on $(-1, 1)$.

1.3   Existence and uniqueness of $\cos \theta$ for $\theta \in [0, \pi]$

Proposition 1.1 (Range of $\mathcal{B}$)**.** *$\mathcal{B}$ maps $[-1, 1]$ onto $[0, \pi]$, with $\mathcal{B}(1) = 0$ and $\mathcal{B}(-1) = \pi$. Moreover, $\mathcal{B}$ is one-to-one on $[-1, 1]$.*

Proof. We know $\mathcal{B}(1) = 2\mathcal{A}(1) = 0$ and $\mathcal{B}(-1) = 2\mathcal{A}(-1) = \pi$. The function $\mathcal{A}$ is continuous on $[-1, 1]$ because it is a sum of continuous functions and an integral with continuous integrand; hence $\mathcal{B}$ is continuous on $[-1, 1]$. By the Intermediate Value Theorem, the image $\mathcal{B}([-1, 1])$ is an interval containing 0 and π, so it contains all of $[0, \pi]$.

For injectivity: since $\mathcal{B}'(x) < 0$ for $-1 < x < 1$, $\mathcal{B}$ is strictly decreasing on $(-1, 1)$, hence one-to-one there; continuity takes care of the endpoints, so $\mathcal{B}$ is one-to-one on $[-1, 1]$. $\square$

We can now define cos and sin in a way that is compatible with the geometric meaning of radians.

Definition 1.2 (Cosine and sine on the interval $[0, \pi]$)**.** For $\theta \in [0, \pi]$, define $\cos \theta$ to be the unique number in $[-1, 1]$ such that

$$\mathcal{B}(\cos \theta) = \theta \qquad (\text{equivalently, } \mathcal{A}(\cos \theta) = \theta/2). \tag{4}$$

Then define

$$\sin \theta = \sqrt{1 - (\cos \theta)^2}. \tag{5}$$

Existence and uniqueness of $\cos \theta$ for $\theta \in [0, \pi]$ follow immediately from Proposition 1.1.

Definition 1.3 (Extending cos and sin to all real numbers)**.** Extend cos and sin from $[0, \pi]$ to $[0, 2\pi]$ by

$$\cos \theta = \cos(2\pi - \theta), \qquad \sin \theta = -\sin(2\pi - \theta) \qquad (\pi \le \theta \le 2\pi),$$

and then extend to all $\mathbb{R}$ by 2π–periodicity:

$$\cos \theta = \cos(\theta \bmod 2\pi), \qquad \sin \theta = \sin(\theta \bmod 2\pi).$$

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2 Why $\cos$ is differentiable (and why $\cos' = -\sin$)

At this point, cos is defined as an *inverse function*: on $(0, \pi)$ it is the inverse of $\mathcal{B}$. So the differentiability of cos on $(0, \pi)$ is a direct application of the Inverse Function Theorem.

2.1 The Inverse Function Theorem we need

Theorem 2.1 (Inverse Function Theorem (interval version))**.** *Let I be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Assume $f'(x) \neq 0$ for all $x \in I$. Then f is one-to-one on I, the inverse function $f^{-1} : f(I) \to I$ exists, and f^{-1} is differentiable on $f(I)$. Moreover, for $y_0 = f(x_0)$ we have*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

2.2 Differentiability of $\cos$ on $(0, \pi)$

Proposition 2.1 (cos is differentiable on $(0, \pi)$)**.** *The function $\cos$ is differentiable on $(0, \pi)$ and*

$$(\cos\theta)' = -\sin\theta \qquad (0 < \theta < \pi).$$

Proof. By Lemma 1.1, $\mathcal{B}$ is differentiable on $(-1, 1)$ with $\mathcal{B}'(x) \neq 0$ there (indeed $\mathcal{B}'(x) < 0$). Therefore Theorem 2.1 applies to $f = \mathcal{B}$ on the interval $(-1, 1)$. In particular, the inverse $\mathcal{B}^{-1}$ exists and is differentiable on $\mathcal{B}((-1, 1)) = (0, \pi)$.

But on $(0, \pi)$, Definition 1.2 says exactly that cos is the inverse of $\mathcal{B}$:

$$\cos\theta = \mathcal{B}^{-1}(\theta) \qquad (0 < \theta < \pi).$$

So cos is differentiable on $(0, \pi)$.

For the derivative, apply the formula from Theorem 2.1:

$$(\cos\theta)' = (\mathcal{B}^{-1})'(\theta) = \frac{1}{\mathcal{B}'(\cos\theta)}.$$

Using (3) with $x = \cos\theta$ gives

$$\mathcal{B}'(\cos\theta) = -\frac{1}{\sqrt{1 - (\cos\theta)^2}} = -\frac{1}{\sin\theta},$$

because for $0 < \theta < \pi$ we have $\sin\theta > 0$ by Definition 1.2. Hence

$$(\cos\theta)' = \frac{1}{-1/\sin\theta} = -\sin\theta,$$

as claimed. $\qquad\square$

2.3 Extending differentiability to the endpoints 0 and π

The inverse function theorem gives differentiability only on the open interval $(0, \pi)$ because $\mathcal{B}'$ blows up at ± 1. Still, cos is differentiable at 0 and π; one neat way to see this is to use the Mean Value Theorem on $\mathcal{B}$.

Proposition 2.2 (cos is differentiable at 0 and π)**.** cos *is differentiable on* $[0, \pi]$ *and*

$$\cos'(0) = 0, \qquad \cos'(\pi) = 0.$$

Proof. We already know cos is differentiable on $(0, \pi)$. We prove differentiability at 0; the proof at π is analogous.

Let $h > 0$ and set $x = \cos h \in (-1, 1)$ (for small h, x is close to 1). By definition of cos, $\mathcal{B}(x) = h$ and $\mathcal{B}(1) = 0$. Apply the Mean Value Theorem to $\mathcal{B}$ on the interval $[x, 1]$ (or $[1, x]$ if you prefer): there exists c between x and 1 such that

$$\mathcal{B}(1) - \mathcal{B}(x) = \mathcal{B}'(c)(1 - x).$$

Since $\mathcal{B}(1) = 0$ and $\mathcal{B}(x) = h$, this becomes

$$-h = \mathcal{B}'(c)(1 - x), \qquad \text{so} \qquad \frac{x - 1}{h} = \frac{1}{\mathcal{B}'(c)}.$$

But $\mathcal{B}'(c) = -1/\sqrt{1 - c^2}$, hence

$$\left| \frac{1}{\mathcal{B}'(c)} \right| = \sqrt{1 - c^2}.$$

As $h \downarrow 0$ we have $x = \cos h \to \cos 0 = 1$, so also $c \to 1$. Therefore $\sqrt{1 - c^2} \to 0$, and thus

$$\lim_{h \downarrow 0} \frac{\cos h - \cos 0}{h} = \lim_{h \downarrow 0} \frac{x - 1}{h} = \lim_{h \downarrow 0} \frac{1}{\mathcal{B}'(c)} = 0.$$

So $\cos'(0) = 0$.

For π, take $h \downarrow 0$ and apply the Mean Value Theorem to $\mathcal{B}$ on $[-1, \cos(\pi - h)]$ to show

$$\lim_{h \downarrow 0} \frac{\cos(\pi - h) - \cos \pi}{-h} = 0,$$

so $\cos'(\pi) = 0$. $\qquad \square$

Remark. Once differentiability is established on $[0, \pi]$, the extension rules in Definition 1.3 propagate differentiability to all of $\mathbb{R}$ (checking the join points π and 2π is routine, and periodicity then handles all $2k\pi$).

3 Compactness: intersections and Heine–Borel

We now switch gears to compactness on the real line. Two different definitions of compactness are in common use:

- *Covering compactness* (also called open-cover compactness);

- *Closed and bounded* (which is *equivalent* to compactness on $\mathbb{R}$ by Heine–Borel).

A lot of confusion disappears once you are comfortable moving between them.

3.1 Definitions

Definition 3.1 (Open cover and covering compactness). Let $K \subseteq \mathbb{R}$. An *open cover* of K is a collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of $\mathbb{R}$ such that $K \subseteq \bigcup_{\alpha \in A} U_\alpha$. We say K is *(covering) compact* if every open cover of K has a *finite* subcover.

Definition 3.2 (Closed and bounded). A set $K \subseteq \mathbb{R}$ is *bounded* if $K \subseteq [-M, M]$ for some $M > 0$. It is *closed* if it contains all its limit points (equivalently, if K^c is open).

Theorem 3.1 (Heine–Borel on $\mathbb{R}$). *A set $K \subseteq \mathbb{R}$ is compact (in the sense of Definition 3.1) if and only if it is closed and bounded.*

We will use Theorem 3.1 as a *destination* in §3.3; first, we address intersections of compact sets.

3.2 Intersection of compact sets is compact (on $\mathbb{R}$)

Proposition 3.1 (Intersections). *Let $\{K_i\}_{i \in I}$ be any collection of compact subsets of $\mathbb{R}$ (finite, countable, or uncountable). Then $\bigcap_{i \in I} K_i$ is compact.*

Proof using Heine–Borel. Assume Theorem 3.1. Each K_i is closed and bounded. An intersection of closed sets is closed, and an intersection of bounded sets is bounded. So $\bigcap_i K_i$ is closed and bounded, hence compact by Heine–Borel. $\qquad\square$

If you want to avoid Heine–Borel and work purely with open covers, it is still true, but it takes a couple of standard lemmas.

Lemma 3.1 (Closed subsets of compact sets are compact). *If $K \subseteq \mathbb{R}$ is compact and $F \subseteq K$ is closed (in $\mathbb{R}$), then F is compact.*

Proof. Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of F. Since F is closed, F^c is open. Then $\{U_\alpha\}_{\alpha \in A} \cup \{F^c\}$ is an open cover of K (because it covers F and also covers $K \setminus F$). By compactness of K, there is a finite subcover of K. Discard F^c if it appears (it covers no point of F), leaving a finite subcover of F. $\qquad\square$

Proposition 3.2 (Intersections via open covers). *If $\{K_i\}_{i \in I}$ are compact subsets of $\mathbb{R}$, then $\bigcap_{i \in I} K_i$ is compact.*

Proof. Pick one index $i_0 \in I$. Then K_{i_0} is compact. Also, each K_i is closed (compact sets in $\mathbb{R}$ are closed; see Lemma 3.3 below), so $\bigcap_i K_i$ is an intersection of closed sets, hence closed. Therefore $\bigcap_i K_i$ is a closed subset of the compact set K_{i_0}, so it is compact by Lemma 3.1. $\quad\square$

3.3 A proof of Heine–Borel

To prove Theorem 3.1, we show two implications.

3.3.1 Compact ⇒ bounded and closed

Lemma 3.2 (Compact sets are bounded). *If $K \subseteq \mathbb{R}$ is compact, then K is bounded.*

Proof. Consider the open sets $U_n = (-n, n)$ for $n \in \mathbb{N}$. If K were unbounded, then $\{U_n\}_{n \in \mathbb{N}}$ would be an open cover of K with no finite subcover (any finite subcollection is contained in $(-N, N)$ for some N). Therefore compactness forces K to be bounded. $\square$

Lemma 3.3 (Compact sets are closed). *If $K \subseteq \mathbb{R}$ is compact, then K is closed.*

Proof. Let $x \in K^c$. For each $y \in K$, define the open interval

$$U_y = \left(y - \frac{|x - y|}{2},\, y + \frac{|x - y|}{2} \right).$$

Then $y \in U_y$ but $x \notin U_y$ (because U_y has radius $|x - y|/2$ around y). So $\{U_y\}_{y \in K}$ is an open cover of K. By compactness, choose $y_1, \ldots, y_n \in K$ such that $K \subseteq U_{y_1} \cup \cdots \cup U_{y_n}$.

Let $r_i = \frac{|x - y_i|}{2} > 0$ and set $\varepsilon = \min\{r_1, \ldots, r_n\} > 0$. We claim that $(x - \varepsilon, x + \varepsilon)$ contains no point of K. Indeed, if $z \in (x - \varepsilon, x + \varepsilon)$ then $|z - x| < \varepsilon \le r_i$ for each i, so

$$|z - y_i| \ge |x - y_i| - |z - x| > 2r_i - r_i = r_i,$$

which implies $z \notin U_{y_i}$ for every i. Hence $z \notin \bigcup_{i=1}^n U_{y_i} \supseteq K$. So $(x - \varepsilon, x + \varepsilon) \subseteq K^c$, proving K^c is open. Therefore K is closed. $\square$

Lemmas 3.2 and 3.3 prove the forward direction of Heine–Borel.

3.3.2 Closed and bounded ⇒ compact

The key step is to prove that a closed interval is compact.

Theorem 3.2 (Closed intervals are compact). *For any $a < b$, the closed interval $[a, b]$ is compact.*

Proof. It suffices to prove $[0, 1]$ is compact; the general case follows by translation and scaling.
Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of $[0, 1]$. Define

$$S = \{x \in [0, 1] : [0, x] \text{ can be covered by finitely many } U_\alpha\}.$$

We will show $S = [0, 1]$.
First, $0 \in S$ because $0 \in U_{\alpha_0}$ for some α_0, so $[0, 0]$ is covered by one open set. Let $s = \sup S$ (which exists because $S \subseteq [0, 1]$ is nonempty and bounded above).
Step 1: $s \in S$. Pick α_* with $s \in U_{\alpha_*}$. Because U_{α_*} is open, there exists $\delta > 0$ such that $(s - \delta, s + \delta) \subseteq U_{\alpha_*}$. By definition of supremum, choose $x \in S$ with $s - \delta/2 < x \le s$. Then $[0, x]$ has a finite subcover, and the remaining part $[x, s]$ lies in U_{α_*}. So $[0, s]$ has a finite subcover, hence $s \in S$.
Step 2: $s = 1$. If $s < 1$, pick $\alpha^\#$ with $s \in U_{\alpha^\#}$ and choose $\delta > 0$ with $(s - \delta, s + \delta) \subseteq U_{\alpha^\#}$. As in Step 1, we can cover $[0, s]$ finitely and then use $U_{\alpha^\#}$ to cover up to $s + \delta/2$. That shows $s + \delta/2 \in S$, contradicting that s is an upper bound. Therefore $s = 1$, i.e. $1 \in S$. So $[0, 1]$ has a finite subcover, proving compactness. $\square$

Now we can finish Heine–Borel.

Proof of Theorem 3.1 (Heine–Borel). We already proved compact $\Rightarrow$ closed and bounded (Lemmas 3.2 and 3.3).

For the converse, let $E \subseteq \mathbb{R}$ be closed and bounded. Boundedness implies $E \subseteq [a, b]$ for some $a < b$. By Theorem 3.2, $[a, b]$ is compact. Since E is closed in $\mathbb{R}$, it is a closed subset of $[a, b]$, hence compact by Lemma 3.1. $\qquad\square$

4 Summary of the "big ideas"

- **Defining functions via integrals.** You can use the Fundamental Theorem of Calculus to show differentiability of integral-defined functions.

- **Inverse functions.** If $f'(x) \neq 0$ on an interval, then f is one-to-one and its inverse is differentiable; moreover $(f^{-1})'(y) = 1/f'(x)$.

- **Closed subsets of compact sets are compact.** This one lemma powers many quick arguments, including intersections of compact sets.

- **Heine–Borel on $\mathbb{R}$.** "Compact" $\Leftrightarrow$ "closed and bounded".