Math 3A03 handout

Monotonicity from a Nonvanishing Derivative: Three Proofs

17 January 2026

NOTE: In Lecture 5 on 16 Jan 2026, we proved the following (as a simple example of the utility of Darboux's theorem):

> If f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$ then f is either increasing or decreasing on the entire interval I.

A student asked whether one could prove this result from the Mean Value Theorem (MVT) without using Darboux's theorem. I thought about this for a few minutes after class and then—with remarkably little effort—coached ChatGPT to write the following LaTeX document. As a matter of interest, there were a few errors in ChatGPT's initial version, which I've corrected. If you notice any errors that I missed, please let me know. –David Earn

Theorem 1. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. If $f'(x) \neq 0$ for all $x \in I$, then f is either strictly increasing on I or strictly decreasing on I.*

We present three proofs:

1. a short proof using Darboux's theorem for derivatives;
2. a proof using Rolle's theorem (a special case of the Mean Value Theorem) plus a basic fact about continuous injective functions on intervals;
3. a proof that *derives Darboux's theorem from the Mean Value Theorem*, and then plugs it into Proof 1.

Background: Mean Value Theorem and Rolle's Theorem

We recall the Mean Value Theorem (MVT): if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's theorem is the special case $f(a) = f(b)$, which forces the secant slope to be 0: there exists $c \in (a, b)$ with $f'(c) = 0$.

Proof 1: Darboux + MVT (the "one-line" strategy)

Darboux's theorem says that derivatives have the intermediate value property.

Proposition 1 (Darboux property for derivatives)**.** *If f is differentiable on an interval I and $x_1 < x_2$ are in I, then for every α between $f'(x_1)$ and $f'(x_2)$ there exists $c \in (x_1, x_2)$ such that $f'(c) = \alpha$.*

Proof of Theorem 1 assuming Proposition 1. If f' takes both positive and negative values on I, then there exist points $p, q \in I$ with $f'(p) > 0$ and $f'(q) < 0$. Let $x_1 = \min\{p, q\}$ and $x_2 = \max\{p, q\}$, so $x_1 < x_2$. Then $f'(x_1)$ and $f'(x_2)$ have opposite signs, hence 0 lies between them. By Proposition 1, there exists $c \in (x_1, x_2)$ such that $f'(c) = 0$, contradicting the hypothesis $f'(x) \neq 0$ for all $x \in I$.

Therefore f' has constant sign on I: either $f'(x) > 0$ for all $x \in I$ or $f'(x) < 0$ for all $x \in I$. Now fix any $a < b$ in I. By the MVT there exists $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Since $b - a > 0$ and $f'(c)$ has the constant sign of f', the difference $f(b) - f(a)$ has that same sign. Thus $f(b) > f(a)$ for all $a < b$ (strictly increasing) or $f(b) < f(a)$ for all $a < b$ (strictly decreasing). ☐

Proof 2: Rolle ⇒ injective ⇒ monotone

This route uses Rolle's theorem and the Intermediate Value Theorem (IVT) for continuous functions.

Lemma 1. *Under the hypotheses of Theorem 1, f is injective on I.*

Proof. Take $a < b$ in I. If $f(a) = f(b)$, then Rolle's theorem gives a point $c \in (a, b)$ with $f'(c) = 0$, contradicting the hypothesis. Hence $f(a) \neq f(b)$ whenever $a \neq b$, so f is injective. ☐

Lemma 2. *If I is an interval and $f : I \to \mathbb{R}$ is continuous and injective, then f is strictly monotone on I.*

Proof. Suppose f is *not* monotone. Then there exist $x_0 < x_1 < x_2$ in I such that $f(x_1)$ is not between $f(x_0)$ and $f(x_2)$. Thus either

$$f(x_1) > \max\{f(x_0), f(x_2)\} \qquad \text{(a peak)},$$

or

$$f(x_1) < \min\{f(x_0), f(x_2)\} \qquad \text{(a valley)}.$$

We treat the peak case; the valley case is identical with inequalities reversed.

Choose a number y with

$$\max\{f(x_0), f(x_2)\} < y < f(x_1).$$

By continuity and the IVT, the equation $f(x) = y$ has a solution $u \in (x_0, x_1)$ and also a solution $v \in (x_1, x_2)$. Then $u \neq v$ but $f(u) = f(v) = y$, contradicting injectivity. Therefore f must be strictly monotone. ☐

Proof of Theorem 1 (Proof 2). Since f is differentiable, it is continuous. By Lemma 1, f is injective. Then Lemma 2 implies f is strictly monotone on I, i.e. either strictly increasing or strictly decreasing. ☐

Why the Mean Value Theorem alone is not (quite) enough

The MVT says that for each pair $a < b$ there exists some $c \in (a, b)$ with

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

But the point c depends on a and b. Knowing only that $f'(c) \neq 0$ does *not* by itself force $f(b) - f(a)$ to have the same sign for *all* pairs $a < b$. To rule out sign changes in f', one needs an additional ingredient that captures an intermediate-value phenomenon (Darboux), or a continuity/topological argument (as in Proof 2).

Proof 3: Derive Darboux from MVT, then use Proof 1

We now show that Proposition 1 itself can be proved from the MVT. The key step is what is sometimes called Fermat's Interior Extremum Theorem (derivative at a local extremum of a differentiable function is zero), which follows from the MVT.

Proposition 2 (Fermat's Interior Extremum Theorem)**.** *Let g be differentiable on an interval containing c. If g has a local maximum or local minimum at c, then $g'(c) = 0$.*

Proof via MVT. Assume g has a local minimum at c (the maximum case is analogous). For small $h > 0$ we have $g(c) \leq g(c + h)$, hence

$$\frac{g(c + h) - g(c)}{h} \geq 0.$$

For small $h > 0$ we also have $g(c) \leq g(c - h)$, hence

$$\frac{g(c) - g(c - h)}{h} \leq 0.$$

Taking limits as $h \downarrow 0$ and using the definition of the derivative gives $g'(c) \geq 0$ and $g'(c) \leq 0$, so $g'(c) = 0$.

(Equivalently: for $h > 0$ and $h < 0$, apply the MVT to g on the intervals with endpoints c and $c + h$ to show the corresponding difference quotients have opposite signs, forcing $g'(c) = 0$ in the limit.) $\square$

Proposition 3 (Darboux's theorem for derivatives (from MVT))**.** *Let f be differentiable on an interval I. If $x_1 < x_2$ are points in I and α lies strictly between $f'(x_1)$ and $f'(x_2)$, then there exists $c \in (x_1, x_2)$ such that $f'(c) = \alpha$.*

Proof. Assume $f'(x_1) < \alpha < f'(x_2)$ (the other order is identical). Define

$$g(x) = f(x) - \alpha x.$$

Then g is differentiable on I and

$$g'(x) = f'(x) - \alpha, \qquad \text{so} \qquad g'(x_1) < 0 < g'(x_2).$$

Since g is continuous on $[x_1, x_2]$, it attains a global minimum on $[x_1, x_2]$.

We claim this minimum cannot occur at an endpoint. Indeed, $g'(x_1) < 0$ means (by the definition of derivative) that for some small $h > 0$,

$$\frac{g(x_1 + h) - g(x_1)}{h} < 0 \quad \Rightarrow \quad g(x_1 + h) < g(x_1),$$

so x_1 is not a minimum point. Similarly, $g'(x_2) > 0$ implies that for some small $h > 0$,

$$\frac{g(x_2) - g(x_2 - h)}{h} > 0 \quad \Rightarrow \quad g(x_2 - h) < g(x_2),$$

so x_2 is not a minimum point.

Therefore g attains its global minimum at some interior point $c \in (x_1, x_2)$. By Fermat's theorem (which follows from the MVT), namely Proposition 2, we have $g'(c) = 0$. Thus $f'(c) = \alpha$. □

Proof of Theorem 1 (Proof 3). Proposition 3 gives the Darboux intermediate value property for f'. Now apply the argument of Proof 1: if f' never vanishes, it cannot change sign, so it is either everywhere positive or everywhere negative; then the MVT shows f is strictly increasing or strictly decreasing on I. □

Summary.

- Proof 1: shortest, using the Darboux property of derivatives plus MVT.
- Proof 2: uses Rolle ⇒ injective and a continuity/IVT argument ⇒ monotone.
- Proof 3: derives the Darboux property from MVT (via Fermat), so Proof 1 is ultimately MVT-based as well.