Mathematics 3A03 Real Analysis I
Winter 2025 Extra Problems
Topic: Mainly Metric Spaces and Construction of $\mathbb{R}$

Miscellaneous

1. Prove that if $f(0) = 0$ and $f(x) = x^2 \sin \frac{1}{x}$ otherwise, then f is differentiable at 0 but not continuous in any interval containing 0.

2. (***Weierstraß function***) Let

$$W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, $b \in \mathbb{N}$ is odd, and $ab > 1$. Prove that W is

(a) continuous everywhere;

(b) differentiable nowhere.
Hint: $\cos(A+B) - \cos A = -2 \sin\left(\frac{B}{2}\right) \sin\left(A + \frac{B}{2}\right)$. Also, recall that to prove that a function $f(h)$ does not converge as $h \to 0$, it is sufficient to find a sequence (h_n) such that $h_n \xrightarrow{n\to\infty} 0$ but the sequence $(f(h_n))$ does not converge as $n \to \infty$.

Metric Spaces

3. Suppose $\mathcal{M}$ is a set, a is a real number, and for all $x, y \in \mathcal{M}$,

$$d_a(x,y) = \begin{cases} 0 & x = y, \\ a & x \neq y. \end{cases}$$

For what values of $a \in \mathbb{R}$ is $(\mathcal{M}, d_a)$ a metric space? Does the answer depend on $\mathcal{M}$?

4. Let $\mathcal{M}$ be a (non-empty) finite set, so $\mathcal{M} = \{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$. What are all the possible metrics on $\mathcal{M}$? (Start with $n = 1$, then 2, then 3 and then try to describe the situation for general n. *Warning:* I'm not aware of a simple algebraic condition in general. If you find one, please let me know.)

5. Let $\mathbb{R}^\infty$ denote the space of sequences of real numbers. For $1 \le p < \infty$, consider the subspace

$$\ell^p \;=\; \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^\infty \;:\; \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

with norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}.$$

Prove that the closed unit ball in ℓ^p,

$$\mathcal{B}_p \;=\; \left\{ x \in \ell^p : \|x\|_p \le 1 \right\},$$

is <u>not</u> compact.

6. Consider ℓ^∞, the space of bounded sequences of real numbers with the sup norm, as in defined in class on slide 69 of the metric spaces lectures. Let $E = (0,1)^\infty$ be the subset of ℓ^∞ (as the example in class) of sequences that are strictly in $(0,1)$. Give examples of points in E° and ∂E. Prove that E is neither open nor closed, and find E° and $\overline{E}$? Is $\overline{E}$ compact?

7. Suppose V is a vector space and $f : V \to \mathbb{R}$. Prove or disprove each of the following statements.

   (a) If f satisfies the parallelogram law,
   $$f(x+y)^2 + f(x-y)^2 \;=\; 2\,f(x)^2 + 2\,f(y)^2, \qquad \forall\, x, y \in V,$$
   then $f(0) = 0$.

   (b) If f satisfies the parallelogram law, then the function g, defined by $g(x) = (f(x))^2$ for all $x \in V$, also satisfies the parallelogram law.

   (c) If f satisfies the parallelogram law, then f is a norm on V.

   (d) If f satisfies the parallelogram law, then $d(x,y) = f(x-y)$ is a metric on V.

8. (a) Prove that the zero function, $f(x) = 0$ for all x, is the unique solution to the integral equation
   $$f(x) = \int_0^x f(t)\,dt \qquad \forall\, x \in [0, \tfrac{1}{2}].$$
   Hint: Consider an appropriate metric space and use the Contraction Mapping Principle.

   (b) Reconsider the result in part (a) but with the interval on which f is defined being $[0,b]$. For which $b \in (0,1] \setminus \{\frac{1}{2}\}$ is the result in part (a) true?

9. Prove or disprove: A contraction map on a metric space is continuous.

Construction of $\mathbb{R}$

10. Recall from class that we defined a ***real number*** to be a subset $\alpha \subseteq \mathbb{Q}$ with the following four properties:

    1. ***downward closed:*** $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$;
    2. ***non-empty:*** $\alpha \neq \varnothing$;
    3. ***non-empty complement:*** $\mathbb{Q} \setminus \alpha \neq \varnothing$, *i.e.*, $\alpha \neq \mathbb{Q}$;
    4. ***no greatest element:*** $\forall x \in \alpha$, $\exists y \in \alpha$ such that $y > x$.

    Assume α and β are real numbers, and define their ***sum*** $\alpha + \beta$ to be

    $$\alpha + \beta \stackrel{\text{def}}{=} \Big\{a + b \,:\, a \in \alpha,\, b \in \beta\Big\}.$$

    Use the formal definition above to show that $\alpha + \beta$ is a real number.

11. Prove that any two norms on $\mathbb{R}^n$ are ***equivalent*** in the sense that $\exists C_1, C_2 > 0$ such that

    $$C_1 \|x\|_a \leq \|x\|_b \leq C_2 \|x\|_a \,, \qquad \forall x \in \mathbb{R}^n.$$

    Hint: Consider the unit sphere (the boundary of the unit ball).

    Does it follow that all norms on $\mathbb{R}^n$ are equivalent? What can you infer about the topology (class of open sets) induced by norms on $\mathbb{R}^n$?

    Denote the space of *eventually zero* sequences by EZ (think "eee-zee"). Are all norms on EZ equivalent? Is this an easy problem?

12. Prove that $\mathbb{R}^n$ is complete wrt (a) the Taxicab norm ($\|\cdot\|_1$) and (b) the Maximum ($\|\cdot\|_\infty$) norm.

    In fact, for any $p \geq 1$, $\mathbb{R}^n$ with norm $\|\cdot\|_p$ is complete wrt to the metric $d(x, y) = \|x - y\|_p$ induced by the norm.

    Is $\mathbb{R}^n$ complete wrt to any norm on $\mathbb{R}^n$?

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