Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 6 (Solutions)

This assignment was **due** on **Tuesday 3 December 2019 at 2:25pm** via crowdmark.

1. Recall from class that we defined a ***real number*** to be a subset $\alpha \subseteq \mathbb{Q}$ with the following four properties:

   1. $\forall x \in \alpha$, if $y \in \mathbb{Q}$ and $y < x$, then $y \in \alpha$;
   2. $\alpha \neq \varnothing$;
   3. $\alpha \neq \mathbb{Q}$;
   4. there is no greatest element in α: $\forall x \in \alpha$, $\exists y \in \alpha$ so that $y > x$.

   Assume α and β are real numbers, and define their ***sum*** $\alpha + \beta$ to be

   $$\alpha + \beta = \{a + b \mid a \in \alpha, b \in \beta\}.$$

   Use the formal definition above to show that $\alpha + \beta$ is a real number.

   Solution: First we must show that $\alpha + \beta \subseteq \mathbb{Q}$. Since $\alpha \subseteq \mathbb{Q}$ and $\beta \subseteq \mathbb{Q}$, for any $a \in \alpha$ and $b \in \beta$, we have $a + b \in \mathbb{Q}$, and hence $\alpha + \beta \subseteq \mathbb{Q}$.

   Property 1: If $x \in \alpha + \beta$ then $x = a + b$ for some $a \in \alpha$ and $b \in \beta$. Suppose $y \in \mathbb{Q}$ and $y < x$. We must show $y \in \alpha + \beta$. To see this, write $y = a + (y - a)$, and note that $y - a \in \mathbb{Q}$ since $y \in \mathbb{Q}$ and $a \in \mathbb{Q}$. But $y < x \implies y - a < x - a$; therefore, since $x - a = b \in \beta$ and β is a real number, $y - a \in \beta$. Thus, $a \in \alpha$, $y - a \in \beta$ and $y = a + (y - a)$, which implies $y \in \alpha + \beta$.

   Property 2: Since α and β are real numbers, we know $\alpha \neq \varnothing$ and $\beta \neq \varnothing$. Therefore, there exist $a \in \alpha$ and $b \in \beta$, which implies $a + b \in \alpha + \beta$, so $\alpha + \beta \neq \varnothing$.

   Property 3: Since $\alpha \neq \mathbb{Q}$ and $\beta \neq \mathbb{Q}$, there exist $a \in \mathbb{Q} \setminus \alpha$ and $b \in \mathbb{Q} \setminus \beta$. Moreover, we must have $a > x$ for all $x \in \alpha$ and $b > y$ for all $y \in \beta$ (otherwise, Property 1 would be violated). Therefore, for all $x \in \alpha$ and all $y \in \beta$, $a + b > x + y$. But any $c \in \alpha + \beta$ can be written $c = x + y$, where $x \in \alpha$ and $y \in \beta$; hence $a + b > c$ for all $c \in \alpha + \beta$. Thus $a + b \in \mathbb{Q} \setminus \alpha + \beta$, which establishes that $\alpha + \beta \neq \mathbb{Q}$.

   Property 4: Let $x \in \alpha + \beta$. Then $x = a + b$ for some $a \in \alpha$ and $b \in \beta$. Since $a \in \alpha$ and α is a real number, there exists $a' \in \alpha$ such that $a' > a$. Similarly, there exists $b' \in \beta$ such that $b' > b$. Let $y = a' + b'$. Then $y \in \alpha + \beta$ since $a' \in \alpha$ and $b' \in \beta$; in addition, since $a' > a$ and $b' > b$, $y = a' + b' > a + b = x$. Therefore, $\alpha + \beta$ has no largest element.

   Thus, $\alpha + \beta$ is a real number. □

2. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)},$$

converges uniformly on $\mathbb{R}$.

Solution: Write

$$f_n(x) = \frac{x}{n(1+nx^2)}.$$

Then the series in question is $\sum_{n=1}^{\infty} f_n(x)$. Note that f_n is twice (in fact, infinitely) differentiable and

$$f_n'(x) = \frac{1-nx^2}{n(1+nx^2)}, \qquad f_n''(x) = -2x\frac{3-nx^2}{(1+nx^2)^3}.$$

Thus,

$$f_n'(x) = 0 \iff x = \pm\frac{1}{\sqrt{n}}, \qquad f_n''(1/\sqrt{n}) < 0, \qquad f_n''(-1/\sqrt{n}) > 0,$$

so $x = 1/\sqrt{n}$ is the global maximum point and $x = -1/\sqrt{n}$ is the global minimum point of $f_n(x)$; moreover,

$$f_n\Big(\pm\frac{1}{\sqrt{n}}\Big) = \pm\frac{1}{2n^{3/2}}.$$

Consequently,

$$|f_n(x)| \le \frac{1}{2n^{3/2}} \qquad \forall x \in \mathbb{R}.$$

Therefore, let

$$M_n = \frac{1}{2n^{3/2}},$$

and note that $\sum_{n=1}^{\infty} M_n$ converges (by the integral test). By the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$. □