Mathematics 3A03 Real Analysis I
Winter 2025 ASSIGNMENT 5 solutions
Topic: Metric Spaces
Participation deadline: **4 April 2025 at 11:25am**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Suppose $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $x_n = (x_{n,1}, \dots, x_{n,N}) \in \mathbb{R}^N$ for each $n \in \mathbb{N}$. Which of the following statements are true for the sequence $(x_n)_{n\in\mathbb{N}}$ in the metric space $(\mathbb{R}^N, \text{Euclidean})$?

■ $x_n \xrightarrow{n\to\infty} x \quad \Longrightarrow \quad x_{n,j} \xrightarrow{n\to\infty} x_j$ in ($\mathbb{R}$,standard) $\forall j = 1, 2, \dots, N$;

■ $x_n \xrightarrow{n\to\infty} x \quad \Longleftarrow \quad x_{n,j} \xrightarrow{n\to\infty} x_j$ in ($\mathbb{R}$,standard) $\forall j = 1, 2, \dots, N$;

☐ (x_n) never converges.

For any $a, b \in \mathbb{R}$, $a^2 \le a^2 + b^2$. Therefore, for each $j \in \{1, 2, \dots, N\}$, we have

$$|x_j| = \sqrt{x_j^2} \le \sqrt{\sum_{i=1}^{N} x_i^2} = \|x\|_2 .$$

Thus, a vector converges in the Euclidean norm exactly when *each* of its individual coordinates converges as real numbers (with the standard distance). ☐

2. Let V be an inner product space, with norm $\|v\| = \sqrt{\langle v, v\rangle}$, $v \in V$. Assume that the sequences $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ are both convergent, $v_n \xrightarrow{n\to\infty} v$ and $w_n \xrightarrow{n\to\infty} w$. Which of the following statements are true?

■ $\langle v_n, w_n\rangle \xrightarrow{n\to\infty} \langle v, w\rangle$, in ($\mathbb{R}$,standard);

☐ $\langle v_n, w_n\rangle \xrightarrow{n\to\infty} \|v\| + \|w\|$, in ($\mathbb{R}$,standard);

☐ $\langle v_n, w_n\rangle$ does not necessarily converge in ($\mathbb{R}$,standard).

The second statement is false: Consider constant sequences $v_n = (1, 0, \dots, 0)\ \forall n$ and $w_n = (0, 0, \dots, 0)\ \forall n$; then $\langle v_n, w_n\rangle = 0\ \forall n$ but $\|v\| + \|w\| = 1 + 0 = 1$.

The first statement is true (so the third statement is false). Given $\varepsilon > 0$ we must show $\exists N \in \mathbb{N}$ such that, for any $n \ge N$, $|\langle v_n, w_n\rangle - \langle v, w\rangle| < \varepsilon$. To that end, observe that, for any n, we have

$$\begin{aligned}
|\langle v_n, w_n\rangle - \langle v, w\rangle| &= |\langle v_n, w_n\rangle - \langle v_n, w\rangle + \langle v_n, w\rangle - \langle v, w\rangle| \\
&= |\langle v_n, (w_n - w)\rangle + \langle (v_n - v)\langle v, w\rangle| \\
&\le |\langle v_n, (w_n - w)\rangle| + |\langle (v_n - v)\langle v, w\rangle| && \text{(by triangle inequality)} \\
&\le \|v_n\|\,\|w_n - w\| + \|w\|\,\|v_n - v\| && \text{(by Cauchy-Schwarz inequality)}
\end{aligned}$$

Here, we can make the second term as small as we like because $\|w\|$ is a fixed non-negative real number and $v_n \xrightarrow{n\to\infty} v$. To make the first term above as small as we like, we need the sequence (v_n) to be bounded. But (v_n) is necessarily bounded (we proved in class that convergent implies bounded for sequences in any metric space). Proceeding formally, we know $\exists C_1 > 0$ for which $\|v_n\| \le C_1$ for all $n \in \mathbb{N}$. Let $C_2 = C_1 + \|w\|$. Since $v_n \xrightarrow{n\to\infty} v$ and $w_n \xrightarrow{n\to\infty} w$, for any $\varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ such that

$$d(v_n, v) = \|v_n - v\| < \frac{\varepsilon}{2C_2}, \quad \forall n \ge N_1,$$
$$d(w_n, w) = \|w_n - w\| < \frac{\varepsilon}{2C_2}, \quad \forall n \ge N_2.$$

Now let $N = \max\{N_1, N_2\}$, so both of the above inequalities are true $\forall n \geq N$. Then, for all $n \geq N$,

$$\begin{aligned} |\langle v_n, w_n \rangle - \langle v, w \rangle| &\leq \|v_n\| \, \|w_n - w\| + \|w\| \, \|v_n - v\| \\ &\leq C_2 \left(\|w_n - w\| + \|v_n - v\|\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, by the definition of the limit, $\langle v_n, w_n \rangle \xrightarrow{n\to\infty} \langle v, w \rangle$ in $\mathbb{R}$. ☐

3. In a metric space $(\mathcal{M}, d)$, any set with no limit points is:

☐ open;

■ closed;

☐ neither open nor closed.

In ($\mathbb{R}$, standard, consider any singleton set $E = \{x\}$. Then E has no limit points and E is not open.

We proved in class that a set in any metric space is closed iff it contains all its limit points. But if it has no limit points then it contains all its limit points! Hence any set without limit points is closed. (It is a good exercise to prove this result without appealing to the theorem proved in class. You will end up essentially proving that theorem.) ☐

4. By $C^n[a, b]$ we mean the space of n-times continuously differentiable functions on the closed interval $[a, b]$, i.e., functions that have n derivatives and that the n^{th} derivative is continuous on $[a, b]$. For $n = 0$, we mean $C[a, b]$. The derivative operator on $C^n[a, b]$ for $n > 0$ is defined by $(D(f))(x) = f'(x)$. The sup norm $\|\cdot\|_\infty$ on $C[a, b]$ is still a norm on $C^n[a, b]$ for $n > 0$ (why?). If $n > 0$, we can also define a "derivative norm",

$$\|f\|_D = \|f\|_\infty + \|f'\|_\infty.$$

Which of the following statements are true?

☐ $D : C^1[a, b] \to C[a, b]$ is a continuous operator under the sup norm;

■ $D : C^1[a, b] \to C[a, b]$ is a continuous operator under the derivative norm;

☐ $D : C^1[a, b] \to C[a, b]$ is not a continuous operator under any norm.

If we have a norm on a vector space then it is still a norm on any subspace. Thus, the sup norm, which is valid on all of $C[a, b]$, is still valid when applied to subspaces with some extra smoothness condition, in particular on $C^n[a, b]$ for any $n > 0$.

$D : C^1[a, b] \to C[a, b]$ is NOT a continuous operator under the SUP norm

We need to show there exists a sequence $\{f_n\} \subset C^1[a, b]$ such that $\|f_n\|_\infty \to 0$, but $\|D(f_n)\|_\infty \not\to 0$. Let $f_n(x) = \frac{1}{n}\sin(nx)$ on $[0, \pi]$. Then $f_n'(x) = \cos(nx)$. Therefore,

$$\|f_n\|_\infty = \frac{1}{n} \xrightarrow{n\to\infty} 0, \quad \text{but} \quad \|f_n'\|_\infty = \|\cos(nx)\|_\infty = 1.$$

Thus, $f_n \to 0$ in the sup norm, but $D(f_n) = f_n' \not\to 0$. Hence the derivative operator is not continuous from $C^1[a, b]$ to $C[a, b]$ in the sup norm.

$D : C^1[a,b] \to C[a,b]$ IS a continuous operator wrt the DERIVATIVE norm

Now suppose $f_n \xrightarrow{n\to\infty} f$ in the derivative norm, *i.e.*,

$$\|f_n - f\|_D \;=\; \|f_n - f\|_\infty + \|f_n' - f'\|_\infty \quad \xrightarrow{n\to\infty} \quad 0.$$

Since norms are non-negative, it follows that $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$ and $\|f_n' - f'\|_\infty \xrightarrow{n\to\infty} 0$. Therefore,

$$\|D(f_n) - D(f)\|_\infty \;=\; \|f_n' - f'\|_\infty \xrightarrow{n\to\infty} 0.$$

Thus, the derivative operator is continuous from $(C^1[a,b], \|\cdot\|_D)$ to $(C[a,b], \|\cdot\|_\infty)$. □

Additional problems

5. (a) Consider $f : \mathcal{M} \to \mathcal{N}$ with domain ($\mathcal{M}$,discrete), and $\mathcal{N}$ any metric space $(\mathcal{N}, \rho)$. Show that any such f is continuous.

   f is continuous iff for any open $U \subseteq \mathcal{N}$, $f^{-1}(U)$ is open in $\mathcal{M}$. But $\mathcal{M}$ is discrete so *every* set in $\mathcal{M}$ is open. Hence any function f defined on $\mathcal{M}$ is continuous, regardless of what the target space is.

   (b) Now suppose $f : \mathcal{M} \to \mathcal{N}$ but the range is ($\mathcal{N}$,discrete), and $(\mathcal{M}, d)$ is a metric space that is not discrete (where "discrete" means $A \subset \mathcal{M}$ is both open and closed iff $A = \mathcal{M}$ or $A = \varnothing$). Show that if f is continuous then f is a constant function.

   Suppose, in order to derive a contradiction, that f is not constant. Then there exist points $x_1, x_2 \in \mathcal{M}$ such that $f(x_1) \neq f(x_2)$. Since $\mathcal{N}$ has the discrete topology, the singleton $\{f(x_1)\}$ is open in $\mathcal{N}$, and so the inverse image

   $$f^{-1}(\{f(x_1)\})$$

   is open in $\mathcal{M}$ by continuity of f. Similarly, $f^{-1}(\{f(x_2)\})$ is open in $\mathcal{M}$. Moreover, since $f(x_1) \neq f(x_2)$, *these inverse images are disjoint.* More generally, for each $y \in \mathcal{N}$, the set

   $$f^{-1}(\{y\})$$

   is open in $\mathcal{M}$, and the collection of these inverse images divides $\mathcal{M}$ into pairwise disjoint open sets. But each $f^{-1}(\{y\})$ is also the inverse image of a closed set (a singleton in any metric space is closed), so it is closed. Hence, every inverse image of a singleton, $f^{-1}(\{y\})$, is both open and closed. Now, since f is not constant, there are at least two disjoint nonempty open-and-closed subsets of $\mathcal{M}$, namely $f^{-1}(\{f(x_1)\})$ and $f^{-1}(\{f(x_2)\})$. This contradicts the assumption that $\mathcal{M}$ is not discrete.

   Therefore, our assumption that f is not constant must be false. Thus, any continuous function from a non-discrete metric space into a discrete space must be constant. □

6. For a set E in a metric space $(\mathcal{M}, d)$, we defined the interior, E°, to be the set of all interior points of E. Show that E° is the largest open set contained in E.

   Hint: Show that E° is open, $E^\circ \subseteq E$, and if U is open and $U \subseteq E$, then $U \subseteq E^\circ$.

   (E° is open) We defined an open set to be a set in which every point is an interior point. So E°—the set of all interior points of E—is open by definition.

   ($E^\circ \subseteq E$) Suppose $x \in E^\circ$. Then x is an interior point of E, *i.e.*, there is a ball $B_\varepsilon(x) \subseteq E$. In particular, $x \in E$. So $E^\circ \subseteq E$.

   (Maximality: U open and $U \subseteq E \implies U \subseteq E^\circ$.) Suppose $x \in U$. Since U is open, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U \subseteq E$. Hence $x \in E^\circ$ by definition. So $U \subseteq E^\circ$. □

7. Consider the metric space $\mathcal{M} = [1,\infty)$ with $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Show that $(\mathcal{M}, d)$ is not complete.

   Hint: Show that the sequence $x_n = n$, $n \in \mathbb{N}$, is a Cauchy sequence that does not converge.

   You should first check that d really is a metric on $\mathcal{M}$. Given that, and defining $(x_n) = (n)$, suppose $m, n \in \mathbb{N}$. WLOG assume $m \geq n$, so $0 \leq m - n \leq m$, which implies $0 \leq \frac{m-n}{m} \leq 1$. Then

$$d(x_n, x_m) = \left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{m-n}{nm}\right| = \left|\frac{m-n}{m}\right| \frac{1}{n} \leq \frac{1}{n},$$

   which can be made arbitrarily small by taking n sufficiently large. Hence (x_n) is a Cauchy sequence wrt d. But $x_n \xrightarrow{n\to\infty} \infty$, so (x_n) does not converge. Therefore, $(\mathcal{M}, d)$ is not complete. □

8. Prove or disprove: An inner product space is necessarily complete.

   The statement is FALSE. An inner product space is not necessarily complete; completeness is an additional property. As noted in class, a complete inner product space is called a ***Hilbert space***. Here are two instructive counterexamples.

   (1) Let $V = C[0,1]$, the space of continuous real-valued functions on $[0,1]$. Define the Euclidean (L^2) inner product via

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx,$$

   with norm $\|f\| = \left(\int_0^1 |f(x)|^2\,dx\right)^{1/2}$. This makes V into an inner product space.

   Now consider the sequence $f_n(x) = x^n$. Each $f_n \in C[0,1]$, and:

$$\|f_n - f_m\|^2 = \int_0^1 |x^n - x^m|^2\,dx = \frac{1}{2m+1} - \frac{2}{m+n+1} + \frac{1}{2n+1} \xrightarrow{n,m\to\infty} 0,$$

   so $\{f_n\}$ is a Cauchy sequence in the L^2 norm.

   However, the pointwise limit of $f_n(x)$ is the function:

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$$

   which is not continuous on $[0,1]$, and hence $f \notin V$. Therefore, $\{f_n\}$ does not converge in V, even though it is Cauchy. So V is not complete. □

   (2) The space ℓ^2 of square-summable sequences is defined as

$$\ell^2 = \left\{x = (x_n) \in \mathbb{R}^\infty : \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\},$$

   with inner product

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n, \qquad \text{and norm} \quad \|x\| = \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{1/2}.$$

   Now let $V \subset \ell^2$ be the space of all sequences with only finitely many nonzero terms:

$$V = \{x = (x_n) \in \ell^2 : x_n = 0 \text{ for all but finitely many } n\}.$$

This space is an inner product space under the same inner product. Now define the sequence:

$$x^{(k)} = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{k}, 0, 0, \dots\right) \in V.$$

Each $x^{(k)} \in V$ since it has only finitely many nonzero terms. Now compute:

$$\|x^{(k)} - x^{(m)}\|^2 = \sum_{n=\min(k,m)+1}^{\max(k,m)} \frac{1}{n^2} \xrightarrow{k,m\to\infty} 0.$$

So $\{x^{(k)}\}$ is Cauchy in the ℓ^2 norm. Its limit is the infinite sequence

$$x = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\right),$$

which belongs to ℓ^2 (since $\sum 1/n^2 < \infty$), but not to V (since it has infinitely many nonzero terms). Hence, $\{x^{(k)}\}$ is a Cauchy sequence in V, but its limit lies outside V, so V is not complete. □

Note: The space V in this example is isomorphic to the space $P[0,1]$ of polynomials discussed in class.

9. Prove or disprove: If $(\mathcal{M}, d)$ is a complete metric space and $F \subset \mathcal{M}$ is a closed subset, then (F, d) is a complete metric space.

The statement is TRUE. Let (x_n) be a Cauchy sequence in F. Since $F \subset \mathcal{M}$ and $\mathcal{M}$ is complete, $\exists x \in \mathcal{M}$ such that $x_n \to x$. But F is closed, so it contains all its limit points. In particular, it must contain x. Hence (F, d) is complete. □

10. Verify the following inequalities, which relate the various p-norms on $\mathbb{R}^n$,

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n\,\|x\|_\infty, \quad \text{and} \quad \|x\|_1 \le \sqrt{n}\,\|x\|_2.$$

Let $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and recall

$$\|x\|_\infty = \max_{1\le j\le n} |x_j|, \quad \|x\|_2 = \left(\sum_{j=1}^n |x_j|^2\right)^{1/2}, \quad \|x\|_1 = \sum_{j=1}^n |x_j|.$$

1. $\|x\|_\infty \le \|x\|_2$

Each coordinate satisfies $|x_j|^2 \le \sum_{k=1}^n |x_k|^2$, so:

$$|x_j| \le \left(\sum_{k=1}^n |x_k|^2\right)^{1/2} = \|x\|_2 \quad \text{for all } j.$$

Hence,

$$\|x\|_\infty = \max_j |x_j| \le \|x\|_2.$$

2. $\|x\|_2 \le \|x\|_1$

For each j, we have $|x_j| \le \|x\|_1$, so $|x_j|^2 \le |x_j| \cdot \|x\|_1$. Summing over j,

$$\sum_{j=1}^n |x_j|^2 \le \sum_{j=1}^n |x_j| \cdot \|x\|_1 = \|x\|_1^2.$$

Taking square roots of both sides, we obtain:

$$\|x\|_2 \le \|x\|_1 .$$

3. $\|x\|_1 \le n \|x\|_\infty$

Since $|x_j| \le \|x\|_\infty$ for each j, summing gives:

$$\|x\|_1 = \sum_{j=1}^{n} |x_j| \le \sum_{j=1}^{n} \|x\|_\infty = n \|x\|_\infty .$$

4. $\|x\|_1 \le \sqrt{n} \|x\|_2$

Apply Cauchy–Schwarz to the vectors $u_j = |x_j|$, $v_j = 1$:

$$\sum_{j=1}^{n} |x_j| \le \left(\sum_{j=1}^{n} |x_j|^2 \right)^{1/2} \cdot \left(\sum_{j=1}^{n} 1^2 \right)^{1/2} = \|x\|_2 \cdot \sqrt{n}.$$

Summarizing, we have

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \le n \|x\|_\infty , \quad \text{and} \quad \|x\|_1 \le \sqrt{n} \|x\|_2 ,$$

as requried. □

Balls in the norm $\|\cdot\|_p$ are often written $B_r^p(x)$. Show that the inequalities above imply that the following sets are *nested:*

$$B_{r/n}^2(x) \subseteq B_{r/n}^\infty(x) \subseteq B_r^1(x) \subseteq B_r^2(x) \subseteq B_r^\infty(x).$$

Let $y \in B_{r/n}^2(x)$, so $\|y - x\|_2 < r/n$. Then

$$\|y - x\|_\infty \le \|y - x\|_2 < r/n,$$

so $y \in B_{r/n}^\infty(x)$, establishing

$$B_{r/n}^2(x) \subseteq B_{r/n}^\infty(x).$$

Next, if $y \in B_{r/n}^\infty(x)$, then

$$\|y - x\|_1 \le n \|y - x\|_\infty < n \cdot \frac{r}{n} = r,$$

so $y \in B_r^1(x)$. Hence

$$B_{r/n}^\infty(x) \subseteq B_r^1(x).$$

Next, suppose $y \in B_r^1(x)$. Then

$$\|y - x\|_2 \le \|y - x\|_1 < r,$$

so $y \in B_r^2(x)$, and we have

$$B_r^1(x) \subseteq B_r^2(x).$$

Finally, if $y \in B_r^2(x)$, then

$$\|y - x\|_\infty \le \|y - x\|_2 < r,$$

so $y \in B_r^\infty(x)$, which implies

$$B_r^2(x) \subseteq B_r^\infty(x),$$

completing the chain of containments. □

11. Prove that a norm $\|\cdot\|$ on a real vector space V is induced by an inner product if and only if it satisfies the **parallelogram law**:

$$\|x+y\|^2 + \|x-y\|^2 \;=\; 2\|x\|^2 + 2\|y\|^2\,, \quad \forall x,y \in V. \tag{1}$$

<u>*Hint:*</u> First, show that any inner product on V induces a norm that satisfies the parallelogram law. Second, given a norm $\|\cdot\|$ on V that satisfies the parallelogram law, show that

$$\langle x,y\rangle \;=\; \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right) \tag{$\heartsuit$}$$

defines an inner product on V, and that the norm induced by this inner product is $\|\cdot\|$.

This is a famous theorem due to Jordan and von Neumann (1935, "On Inner Products in Linear, Metric Spaces", *Annals of Mathematics* **36**(3), 719–723). They took the scalar field to be $\mathbb{C}$ rather than $\mathbb{R}$.

<u>*Proof:*</u> We aim to prove "a norm is induced by an inner product" $\iff$ "the norm satisfies the parallelogram law".

$(\implies)$

Given an inner product $\langle\cdot,\cdot\rangle$ on a vector space V, and the induced norm $\|\cdot\|$, for any $x,y\in V$ we have

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y,x+y\rangle + \langle x-y,x-y\rangle && \text{(2a)}\\
&= \Big(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\Big) + \Big(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle\Big) && \text{(2b)}\\
&= 2\langle x,x\rangle + 2\langle y,y\rangle && \text{(2c)}\\
&= 2\|x\|^2 + 2\|y\|^2\,, && \text{(2d)}
\end{aligned}$$

as required.

$(\impliedby)$

First, note that (changing a sign in the calculation above) given an inner product $\langle\cdot,\cdot\rangle$, and the induced norm $\|\cdot\|$, for any $x,y\in V$, we have

$$\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 &= \langle x+y,x+y\rangle - \langle x-y,x-y\rangle && \text{(3a)}\\
&= \Big(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\Big) - \Big(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle\Big) && \text{(3b)}\\
&= 2\langle x,y\rangle + 2\langle y,x\rangle && \text{(3c)}\\
&= 4\langle x,y\rangle\,. && \text{(3d)}
\end{aligned}$$

Solving for $\langle x,y\rangle$, we see that if a norm is induced by an inner product, then the inner product can be expressed using the norm via ($\heartsuit$). Thus, if there is an inner product $\langle\cdot,\cdot\rangle$ that induces a given norm $\|\cdot\|$, then the inner product <u>must</u> be given by ($\heartsuit$).

We now need to show that, given any norm, if we define $\langle\cdot,\cdot\rangle$ via ($\heartsuit$) then $\langle\cdot,\cdot\rangle$ is actually an inner product if and only if the norm satisfies the parallelogram law (1). To that end, we examine each of the axioms of an inner product, assuming $\langle\cdot,\cdot\rangle$ is defined from a given norm $\|\cdot\|$ via ($\heartsuit$). *This is <u>not</u> an easy problem. If you solved it, you should be proud!*

So, suppose $\|\cdot\|$ is a norm on V that satisfies the parallelogram law (1), and that $\langle\cdot,\cdot\rangle$ is a candidate inner product defined by ($\heartsuit$).

(a) ***Symmetry***.

From the definition,

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right). \tag{4}$$

Swapping x and y, we see that:

$$\langle y, x\rangle = \frac{1}{4}\left(\|y+x\|^2 - \|y-x\|^2\right) = \langle x, y\rangle. \tag{5}$$

Thus, the inner product is symmetric. □

(b) ***Positive Definiteness***.

By definition,

$$\langle x, x\rangle = \frac{1}{4}\left(\|x+x\|^2 - \|x-x\|^2\right) = \frac{1}{4}\left(\|2x\|^2 - 0\right) = \frac{1}{4}\left(4\|x\|^2\right) = \|x\|^2. \tag{6}$$

Since norms are always nonnegative and zero if and only if $x = 0$, we conclude $\langle x, x\rangle = 0$ if and only if $x = 0$.

Thus, $\langle x, x\rangle$ is positive definite. □

(c) ***Linearity in the First Argument***.

This is the very challenging part. We will break it into two pieces: $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle ax, z\rangle = a\langle x, z\rangle$. Both are tricky, but—perhaps suprisingly—the second requires more analytical creativity.

To begin with, it is worth making a note of what the paralleogram law (1) implies if one of the two vectors is the zero vector. If $x = 0$, then (1) states that $\|y\|^2 + \|-y\|^2 = 2\|y\|^2$, and hence[1]

$$\|-y\|^2 = \|y\|^2 \quad \forall y \in V. \tag{7}$$

Consequently, if we insert $x = 0$ in (♡), then we we find

$$\langle 0, y\rangle = 0 \quad \forall y \in V. \tag{8}$$

This may seem an obvious fact, but remember that we do <u>not</u> yet know that $\langle\cdot,\cdot\rangle$ is an inner product. All we know is that it is a function of two variables defined by (♡) and that the norm satisfies (1).

Now, the parallelogram law (1) is presumed to hold for all $x, y \in V$. So, it is true if we write x and y as any linear combinations of other vectors in V. Doing this judiciously will get us where we want to go. Given any $u, v, w \in V$, if we let $x = u + v$ and $y = w$ then (1) states that

$$\|u+v+w\|^2 + \|u+v-w\|^2 = 2\|u+v\|^2 + 2\|w\|^2. \tag{9}$$

Similarly, if we let $x = u - v$ and $y = w$ then (1) states that

$$\|u-v+w\|^2 + \|u-v-w\|^2 = 2\|u-v\|^2 + 2\|w\|^2. \tag{10}$$

If we now subtract Eq. (10) from Eq. (9) we obtain

$$\|u+v+w\|^2 + \|u+v-w\|^2 - \|u-v+w\|^2 - \|u-v-w\|^2 = 2\|u+v\|^2 - 2\|u-v\|^2. \tag{11}$$

[1]Note that any norm on V satisfies $\|ay\| = |a|\,\|y\|$ for all $a \in \mathbb{R}$ by definition, so $\|-y\| = \|y\|$, and hence $\|-y\|^2 = \|y\|^2$, regardless of whether the parallelogram law (1) holds in V. But it is interesting that $\|-y\|^2 = \|y\|^2$ follows directly from the parallelogram law, which does not itself imply $\|\cdot\|$ is a norm. For example, if we define a function $\|\cdot\|$ on $\mathbb{R}^n$ via $\|x\| = x_1$ (the first component of x) then $\|\cdot\|$ satisfies the parallelogram law (1) but is <u>not</u> a norm, since $\|-x\| = -x_1 < 0$ for $x_1 > 0$, and hence fails to be non-negative.

Recalling the definition of the candidate inner product (♡), the RHS of Eq. (11) is

$$2\|u+v\|^2 - 2\|u-v\|^2 \;=\; 8\langle u,v\rangle \;, \tag{12}$$

and the LHS of Eq. (11) is

$$\begin{aligned}
&\|(u+v)+w\|^2 + \|(u+v)-w\|^2 - \|(u-v)+w\|^2 - \|(u-v)-w\|^2 && \text{(13a)}\\
&\quad = \|(u+w)+v\|^2 + \|(u-w)+v\|^2 - \|(u+w)-v\|^2 - \|(u-w)-v\|^2 && \text{(13b)}\\
&\quad = \|(u+w)+v\|^2 - \|(u+w)-v\|^2 + \|(u-w)+v\|^2 - \|(u-w)-v\|^2 && \text{(13c)}\\
&\quad = 4\langle u+w,v\rangle + 4\langle u-w,v\rangle \;. && \text{(13d)}
\end{aligned}$$

Therefore, Eq. (11) can be written

$$\langle u+w,v\rangle + \langle u-w,v\rangle \;=\; 2\langle u,v\rangle \;. \tag{14}$$

It would help us if we could replace $2\langle u,v\rangle$ with $\langle 2u,v\rangle$ in this equation, but we don't (yet) know that scalar multiples can be brought inside the first argument of $\langle\cdot,\cdot\rangle$. Nevertheless, we can see that this is true if the scalar multiple happens to be 2, by considering the special case of Eq. (14) with $u=w$, which yields

$$\langle 2u,v\rangle + \langle 0,v\rangle \;=\; 2\langle u,v\rangle \;, \tag{15}$$

and, recalling Eq. (8), this becomes

$$\langle 2u,v\rangle \;=\; 2\langle u,v\rangle \;. \tag{16}$$

Thus, in Eq. (14), we can indeed replace $2\langle u,v\rangle$ with $\langle 2u,v\rangle$ to obtain

$$\langle u+w,v\rangle + \langle u-w,v\rangle \;=\; \langle 2u,v\rangle \;. \tag{17}$$

This is true for <u>any</u> $u,v,w\in V$, so, given any $x,y,z\in V$ (unrelated to any particular x,y,z we started with), if we insert

$$\begin{aligned}
u &= \frac{1}{2}(x+y),\\
v &= z,\\
w &= \frac{1}{2}(x-y),
\end{aligned} \tag{18}$$

in Eq. (17) we obtain

$$\langle x,z\rangle + \langle y,z\rangle \;=\; \langle x+y,z\rangle \;, \quad \forall x,y,z\in V, \tag{19}$$

as required.

Now the really tricky part: we need to prove that for any $x,y\in V$ and any $a\in\mathbb{R}$, $\langle ax,y\rangle = a\langle x,y\rangle$. Our first step is to observe that Eq. (16) is, in fact, a special case of what we are now aiming to prove, namely the case $a=2$. Using Eq. (16) and Eq (19), it follows that

$$\langle 3u,v\rangle \;=\; \langle 2u+u,v\rangle \;=\; \langle 2u,v\rangle + \langle u,v\rangle \;=\; 2\langle u,v\rangle + \langle u,v\rangle \;=\; 3\langle u,v\rangle\,. \tag{20}$$

Similarly, (formally by induction) we have

$$\langle nu,v\rangle \;=\; n\langle u,v\rangle \qquad \forall n\in\mathbb{N}. \tag{21}$$

Now consider a rational number $\frac{m}{n}$, with $m,n\in\mathbb{N}$. We have

$$\begin{aligned}
&\left\langle \frac{m}{n}u,v\right\rangle \;=\; m\left\langle \frac{1}{n}u,v\right\rangle && \text{(22a)}\\
\Longrightarrow\quad & n\left\langle \frac{m}{n}u,v\right\rangle \;=\; nm\left\langle \frac{1}{n}u,v\right\rangle \;=\; m\langle u,v\rangle && \text{(22b)}\\
\Longrightarrow\quad & \left\langle \frac{m}{n}u,v\right\rangle \;=\; \frac{m}{n}\langle u,v\rangle\,, && \text{(22c)}
\end{aligned}$$

so $\langle au, v\rangle = a\langle u, v\rangle$ for any positive rational number. If we now observe, using Eqs. (8) and (19), that

$$0 \;=\; \langle 0, v\rangle \;=\; \langle u + (-u), v\rangle \;=\; \langle u, v\rangle + \langle -u, v\rangle \;, \tag{23}$$

then we have

$$\langle -u, v\rangle \;=\; -\langle u, v\rangle \quad \forall u, v \in V, \tag{24}$$

which then implies that, in fact, $\langle au, v\rangle = a\langle u, v\rangle$ for all $a \in \mathbb{Q}$.
Another way of saying this is that, for any $u, v \in V$, the function

$$f(a) \;=\; \langle au, v\rangle - a\langle u, v\rangle \tag{25}$$

satisfies $f(a) = 0$ for all $a \in \mathbb{Q}$. If we can now show f is a continuous function of a for all $a \in \mathbb{R}$, then it will follow that $f(a) = 0$ for all $a \in \mathbb{R}$. To that end, first recall that in any normed vector space V,

$$\left|\|x\| - \|y\|\right| \;\leq\; \|x - y\| \qquad \forall\, x, y \in V \tag{26}$$

(which you can prove by noting that $\|x\| = \|(x-y) + y\|$ and using the triangle inequality). Consequently, if $\langle\cdot,\cdot\rangle$ is defined by (♡) then

$$\begin{aligned}
|\langle x, y\rangle| \;&=\; \frac{1}{4}\Big|\, \|x+y\|^2 - \|x-y\|^2 \,\Big| && (27a)\\
&=\; \frac{1}{4}\Big|\big(\,\|x+y\| - \|x-y\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| && (27b)\\
&=\; \frac{1}{4}\Big|\big(\,\|x+y\| - \|-(x-y)\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| \quad \text{from (7)} && (27c)\\
&\leq\; \frac{1}{4}\Big|\big(\,\|(x+y) + (x-y)\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| \quad \text{from (26)} && (27d)\\
&=\; \frac{1}{4}\Big|\big(\,\|2x\|\,\big)\big(\,\|x+y\| + \|x-y\|\,\big)\Big| && (27e)\\
&\leq\; \frac{1}{4}\Big|\big(\,\|2x\|\,\big)\big(\,\|x\| + \|y\| + \|x\| + \|y\|\,\big)\Big| \quad \text{triangle inequality} && (27f)\\
&=\; \frac{1}{4}\Big|\big(2\,\|x\|\,\big)\big(2\,\|x\| + 2\,\|y\|\,\big)\Big| && (27g)
\end{aligned}$$

Cancelling constants, we have

$$|\langle x, y\rangle| \;\leq\; \|x\|\,\big(\,\|x\| + \|y\|\,\big)\,. \tag{28}$$

Now suppose $a_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$, and $a_n \xrightarrow{n\to\infty} a \in \mathbb{R}$. We need to show that $f(a_n) \xrightarrow{n\to\infty} f(a)$ to establish that f is continuous on $\mathbb{R}$. Therefore, consider

$$\begin{aligned}
|f(a_n) - f(a)| \;&=\; \Big|\langle a_n u, v\rangle - a_n\langle u, v\rangle - \big(\,\langle au, v\rangle - a\langle u, v\rangle\,\big)\Big| && (29a)\\
&=\; \Big|\langle a_n u, v\rangle - \langle au, v\rangle - \big(a_n\langle u, v\rangle - a\langle u, v\rangle\,\big)\Big| && (29b)\\
&=\; |\langle a_n u, v\rangle - \langle au, v\rangle - (a_n - a)\langle u, v\rangle| && (29c)\\
&=\; |\langle a_n u, v\rangle + \langle -au, v\rangle - (a_n - a)\langle u, v\rangle| \quad \text{from (24)} && (29d)\\
&=\; |\langle a_n u - au, v\rangle - (a_n - a)\langle u, v\rangle| \quad \text{from (19)} && (29e)\\
&=\; |\langle (a_n - a)u, v\rangle - (a_n - a)\langle u, v\rangle| && (29f)\\
&\leq\; |\langle (a_n - a)u, v\rangle| + |(a_n - a)\langle u, v\rangle| && (29g)\\
&=\; |\langle (a_n - a)u, v\rangle| + |a_n - a|\,|\langle u, v\rangle| && (29h)\\
&\leq\; \|(a_n - a)u\|\,(\|(a_n - a)u\| + \|v\|) + |a_n - a|\,|\langle u, v\rangle| \quad \text{from (28)} && (29i)\\
&=\; |a_n - a|\,\Big(\,\|u\|\,\big(\,|a_n - a|\,\|u\| + \|v\| + |\langle u, v\rangle|\,\big)\Big) && (29j)\\
&\xrightarrow{n\to\infty}\; 0. && (29k)
\end{aligned}$$

Thus, $f(a_n) \xrightarrow{n\to\infty} f(a)$. Since $a \in \mathbb{R}$ was arbitrary, f is continuous on $\mathbb{R}$. Therefore, $f(a) = 0$ for all $a \in \mathbb{R}$, and hence, by definition (25), $\langle au, v\rangle = a\langle u, v\rangle$ for all $a \in \mathbb{R}$, as required. □

12. For each of the real vector spaces $\mathbb{R}^n$, ℓ^p, and $C[a,b]$, prove that the only p-norm that is induced by an inner product is the Euclidean norm.

Hint: Show that a p-norm satisfies the parallelogram law if and only if $p=2$.

In $\mathbb{R}^n$, the p-norm is

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \tag{30}$$

so the parallelogram law (1) states

$$\left(\sum_{i=1}^{n} |x_i+y_i|^p\right)^{2/p} + \left(\sum_{i=1}^{n} |x_i-y_i|^p\right)^{2/p} = 2\left(\sum_{i=1}^{n} |x_i|^p\right)^{2/p} + 2\left(\sum_{i=1}^{n} |y_i|^p\right)^{2/p}. \tag{31}$$

If this law holds for all $x, y \in \mathbb{R}^n$, then it holds in particular for $x = e_1 = (1,0,0,\ldots,0)$ and $y = e_2 = (0,1,0,\ldots,0)$. Inserting these specific values of x and y into Eq (31), we have

$$\|x+y\|_p^2 = (1^p+1^p)^{2/p} = 2^{2/p}, \tag{32}$$

$$\|x-y\|_p^2 = (1^p+1^p)^{2/p} = 2^{2/p}, \tag{33}$$

so the LHS of the parallelogram law (31) is

$$2^{2/p} + 2^{2/p} = 2\cdot 2^{2/p} = 2^{1+2/p}, \tag{34}$$

and the RHS is

$$2\|x\|_p^2 + 2\|y\|_p^2 = 2(1^p)^{2/p} + 2(1^p)^{2/p} = 2+2 = 4. \tag{35}$$

If the identity (31) holds, then

$$2^{1+2/p} = 2^2 \quad\iff\quad 1+\frac{2}{p} = 2 \quad\iff\quad p=2, \tag{36}$$

so the only value of p for which $\|\cdot\|_p$ could possibly satisfy the parallelogram law (31) is $p=2$.

To complete the proof, we must now show that for $p=2$ the parallelogram law (31) holds for <u>all</u> $x, y \in \mathbb{R}^n$, not just $x=e_1$ and $y=e_2$. With $p=2$, the LHS of the law (31) is

$$\sum_{i=1}^{n} |x_i+y_i|^2 + \sum_{i=1}^{n} |x_i-y_i|^2 = \sum_{i=1}^{n} (x_i+y_i)^2 + \sum_{i=1}^{n} (x_i-y_i)^2 \tag{37a}$$

$$= \sum_{i=1}^{n} (x_i^2 + \cancel{2x_iy_i} + y_i^2) + \sum_{i=1}^{n} (x_i^2 - \cancel{2x_iy_i} + y_i^2) \tag{37b}$$

$$= 2\sum_{i=1}^{n} |x_i|^2 + 2\sum_{i=1}^{n} |y_i|^2, \tag{37c}$$

which is the RHS of the law (31), as required.

Thus, given the Jordan-von Neumann theorem (problem 11), we have confirmed that a p-norm on $\mathbb{R}^n$ is induced by an inner product if and only if $p=2$, the Euclidean case. □