Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 5 (Solutions)

This assignment was **due** on **Thursday 21 November 2019 at 2:25pm** via crowdmark.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. A function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all x, and odd if $f(-x) = -f(x)$ for all x. Suppose f is differentiable. Prove, directly from the definition of the derivative, that (a) if f is even then f' is odd and (b) if f is odd then f' is even.

   Solution: If f is differentiable and even then

$$\begin{aligned}
f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(-(x-h))-f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(x-h)-f(x)}{h} && f \text{ is even} \\
&= \lim_{h\to 0} \frac{f(x+(-h))-f(x)}{h} \\
&= \lim_{h\to 0} -\frac{f(x+(-h))-f(x)}{-h} \\
&= -\lim_{h\to 0} \frac{f(x+(-h))-f(x)}{-h} \\
&= -\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && h \text{ can be either sign} \\
&= -f'(x)
\end{aligned}$$

   so f' is odd. Similarly, if f is differentiable and odd then

$$\begin{aligned}
f'(-x) &= \lim_{h\to 0} \frac{f(-x+h)-f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(-(x-h))-f(-x)}{h} \\
&= \lim_{h\to 0} \frac{-f(x-h)+f(x)}{h} && f \text{ is odd} \\
&= \lim_{h\to 0} \frac{-f(x+(-h))-(-f(x))}{h} \\
&= \lim_{h\to 0} -\frac{f(x+(-h))-f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+(-h))-f(x)}{-h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && h \text{ can be either sign} \\
&= f'(x)
\end{aligned}$$

so f' is even. □

2. Establish that the hypotheses of Rolle's Theorem are necessary by constructing functions f that have the following properties, but for which it is not true that there exists $x \in (a, b)$ such that $f'(x) = 0$. In each case, state which hypothesis of Rolle's Theorem is not satified in your example.

   (a) f is continuous on $[a, b]$ and differentiable on (a, b);
   Solution: $a = 1,\ b = 2,\ f(x) = x$
   $(f(1) \neq f(2))$.

   (b) f is continuous on $[a, b]$ and $f(a) = f(b)$;
   Solution: $a = -1,\ b = 1,\ f(x) = |x|$
   (f is not differentiable on $(-1, 1)$, since it has no derivative at 0).

   (c) f is differentiable on (a, b) and $f(a) = f(b)$.
   Solution: $a = 0,\ b = 1,\ f(x) = \begin{cases} f(x) = \frac{1}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$
   (f is not continuous on $[0, 1]$, since it is discontinuous at 0).

3. Prove that if $a < b$ and f is integrable on the closed interval $[a, b]$ then f is necessarily integrable on any closed subinterval of $[a, b]$.

   Solution: Suppose $a < c < d < b$. We will show that f is integrable on each of the subintervals, $[a, c]$, $[c, d]$, $[d, b]$ (which covers all possible types of closed subintervals). Since f is integrable on $[a, b]$, given any $\varepsilon > 0$ we can find a partition $P = \{t_0, \dots, t_n\}$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

   Now let Q be the partition of $[a, b]$ that contains all the points of P and (if they are not already in P) the points c and d. Since $P \subseteq Q$, it follows that

$$U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) < \varepsilon\,.$$

   Since Q contains c and d, we can break it up in the three parts, $Q = Q_1 \cup Q_2 \cup Q_3$, where (for some $j, k \in \mathbb{N}$)

$$\begin{aligned} Q_1 &= \{a, t_1, \dots, t_{j-1}, c\}\,, \\ Q_2 &= \{c, t_{j+1}, \dots, t_{k-1}, d\}\,, \\ Q_3 &= \{d, t_{k+1}, \dots, t_{n-1}, b\}\,. \end{aligned}$$

   Consequently,

$$\begin{aligned} U(f, Q) &= U(f, Q_1) + U(f, Q_2) + U(f, Q_3)\,, \\ L(f, Q) &= L(f, Q_1) + L(f, Q_2) + L(f, Q_3)\,, \end{aligned}$$

   and hence

$$U(f, Q) - L(f, Q) = \big[U(f, Q_1) - L(f, Q_1)\big] + \big[U(f, Q_2) - L(f, Q_2)\big] + \big[U(f, Q_3) - L(f, Q_3)\big].$$

But each of the terms in square brackets is non-negative, and hence each of these terms must itself be less than ε. Thus, we have found partitions (Q_1, Q_2 and Q_3) of $[a,c]$, $[c,d]$ and $[d,b]$, respectively, that ensure the difference between the upper and lower sums of f for Q_i is less than ε, *i.e.,* f is, in fact, integrable on each of the three subintervals. $\square$

4. Define $f:\mathbb{R}\to\mathbb{R}$ via $f(x)=x$ if $x\in\mathbb{Q}$ and $f(x)=0$ if $x\notin\mathbb{Q}$.

   (a) Let P be any partition of $[0,1]$. Find $L(f,P)$.

   Solution: Regardless of how $[0,1]$ is partitioned, every subinterval $[t_{i-1},t_i]$ contains irrational numbers, hence $m_i=\inf\{f(x):x\in[t_{i-1},t_i]\}=0$ for all i. Consequently, $L(f,P)=0$ for any partition P.

   (b) Find $\inf\{U(f,P):P \text{ a partition of } [0,1]\}$.

   Solution: Let $P=\{t_0,t_1,\ldots,t_n\}$ be a partition of $[0,1]$ (so $t_0=0$ and $t_n=1$). For any i, if $t_i\in\mathbb{Q}$ then $M_i=\sup\{f(x):x\in[t_{i-1},t_i]\}=f(t_i)=t_i$. On the other hand, if $t_i\notin\mathbb{Q}$ then—since $\mathbb{Q}$ is dense in $\mathbb{R}$—for all $\varepsilon>0$ there exists δ such that $0<\delta<\varepsilon$ and $t_i-\delta\in\mathbb{Q}$, and hence $f(t_i-\delta)=t_i-\delta>t_i-\varepsilon$. Hence $M_i=\sup\{f(x):x\in[t_{i-1},t_i]\}=t_i$. Thus, for any partition P of $[0,1]$ we have

$$U(f,P)=\sum_{i=1}^{n}M_i(t_i-t_{i-1})=\sum_{i=1}^{n}t_i(t_i-t_{i-1})\,. \qquad (\heartsuit)$$

   In particular, we can write

$$\begin{aligned}
U(f,P)&=\sum_{i=1}^{n}M_i(t_i-t_{i-1})=\sum_{i=1}^{n}t_i(t_i-t_{i-1})\\
&=\sum_{i=1}^{n}[t_i-t_{i-1}+t_{i-1}](t_i-t_{i-1})=\sum_{i=1}^{n}[(t_i-t_{i-1})^2+t_{i-1}(t_i-t_{i-1})]\\
&\geq\sum_{i=1}^{n}\big[\tfrac{1}{2}(t_i-t_{i-1})^2+t_{i-1}(t_i-t_{i-1})\big] \qquad \text{(area of trapezoid)}\\
&=\sum_{i=1}^{n}\big[\tfrac{1}{2}(t_i^2-2t_it_{i-1}+t_{i-1}^2)+t_{i-1}t_i-t_{i-1}^2\big]\\
&=\sum_{i=1}^{n}\big[\tfrac{1}{2}(t_i^2+t_{i-1}^2)-t_{i-1}^2\big]=\sum_{i=1}^{n}\big[\tfrac{1}{2}(t_i^2-t_{i-1}^2)\big]=\frac{1}{2}\sum_{i=1}^{n}\big(t_i^2-t_{i-1}^2\big)\\
&=\frac{1}{2}\big[\big(t_n^2-t_{n-1}^2\big)+\big(t_{n-1}^2-t_{n-2}^2\big)+\cdots+\big(t_1^2-t_0^2\big)\big]\\
&=\frac{1}{2}\big(t_n^2-t_0^2\big)\\
&=\frac{1}{2}\big(1^2-0^2\big)=\frac{1}{2}
\end{aligned}$$

   Thus, $U(f,P)\geq\frac{1}{2}$ for any partition P of $[0,1]$. It therefore follows that

$$\inf\big\{U(f,P):P \text{ a partition of } [0,1]\big\}\geq\frac{1}{2}\,.$$

Note that the side comment "area of trapezoid" is meant to explain where the idea came from, but the argument itself is purely algebraic and does not depend on any picture.

We must now show that $\inf\{U(f,P) : P \text{ a partition of } [0,1]\} \le \frac{1}{2}$. To that end, consider the particular partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\}$, *i.e.*, $t_i = \frac{i}{n}$. From ($\heartsuit$), we have

$$\begin{aligned} U(f,P_n) &= \sum_{i=1}^{n} t_i(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\Big(\frac{i}{n} - \frac{i-1}{n}\Big) \\ &= \sum_{i=1}^{n} \frac{i}{n}\Big(\frac{1}{n}\Big) = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}\,, \end{aligned}$$

which implies

$$\inf\{U(f,P_n) : n \in \mathbb{N}\} = \frac{1}{2}\,,$$

and hence

$$\inf\{U(f,P) : P \text{ a partition of } [0,1]\} \le \frac{1}{2}\,.$$

Consequently, $\inf\{U(f,P)\} = \dfrac{1}{2}$. □

(c) Is f integrable on $[0,1]$?

Solution: No. $\sup\{L(f,P)\} = 0 \ne \frac{1}{2} = \inf\{U(f,P)\}$. □

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