Mathematics 3A03 Real Analysis I
Winter 2025 ASSIGNMENT 5
Topic: Metric Spaces
Participation deadline: **4 April 2025 at 11:25am**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Suppose $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $x_n = (x_{n,1}, \dots, x_{n,N}) \in \mathbb{R}^N$ for each $n \in \mathbb{N}$. Which of the following statements are true for the sequence $(x_n)_{n\in\mathbb{N}}$ in the metric space $(\mathbb{R}^N, \text{Euclidean})$?

   ☐ $x_n \xrightarrow{n\to\infty} x \quad \Longrightarrow \quad x_{n,j} \xrightarrow{n\to\infty} x_j$ in $(\mathbb{R}$,standard$)$ $\forall j = 1, 2, \dots, N$;

   ☐ $x_n \xrightarrow{n\to\infty} x \quad \Longleftarrow \quad x_{n,j} \xrightarrow{n\to\infty} x_j$ in $(\mathbb{R}$,standard$)$ $\forall j = 1, 2, \dots, N$;

   ☐ (x_n) never converges.

2. Let V be an inner product space, with norm $\|v\| = \sqrt{\langle v, v\rangle}$, $v \in V$. Assume that the sequences $(v_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}}$ are both convergent, $v_n \xrightarrow{n\to\infty} v$ and $w_n \xrightarrow{n\to\infty} w$. Which of the following statements are true?

   ☐ $\langle v_n, w_n\rangle \xrightarrow{n\to\infty} \langle v, w\rangle$, in $(\mathbb{R}$,standard$)$;

   ☐ $\langle v_n, w_n\rangle \xrightarrow{n\to\infty} \|v\| + \|w\|$, in $(\mathbb{R}$,standard$)$;

   ☐ $\langle v_n, w_n\rangle$ does not necessarily converge in $(\mathbb{R}$,standard$)$.

3. In a metric space $(\mathcal{M}, d)$, any set with no limit points is:

   ☐ open;

   ☐ closed;

   ☐ neither open nor closed.

4. By $C^n[a, b]$ we mean the space of n-times continuously differentiable functions on the closed interval $[a, b]$, i.e., functions that have n derivatives and that the n^{th} derivative is continuous on $[a, b]$. For $n = 0$, we mean $C[a, b]$. The derivative operator on $C^n[a, b]$ for $n > 0$ is defined by $(D(f))(x) = f'(x)$. The sup norm $\|\cdot\|_\infty$ on $C[a, b]$ is still a norm on $C^n[a, b]$ for $n > 0$ (why?). If $n > 0$, we can also define a "derivative norm",

   $$\|f\|_D = \|f\|_\infty + \|f'\|_\infty.$$

   Which of the following statements are true?

   ☐ $D : C^1[a, b] \to C[a, b]$ is a continuous operator under the sup norm;

   ☐ $D : C^1[a, b] \to C[a, b]$ is a continuous operator under the derivative norm;

   ☐ $D : C^1[a, b] \to C[a, b]$ is not a continuous operator under any norm.

Additional problems

5. (a) Consider $f : \mathcal{M} \to \mathcal{N}$ with domain ($\mathcal{M}$,discrete), and $\mathcal{N}$ any metric space $(\mathcal{N}, \rho)$. Show that any such f is continuous.

   (b) Now suppose $f : \mathcal{M} \to \mathcal{N}$ but the range is ($\mathcal{N}$,discrete), and $(\mathcal{M}, d)$ is a metric space that is not discrete (where "discrete" means $A \subset \mathcal{M}$ is both open and closed iff $A = M$ or $A = \varnothing$). Show that f is a constant function.

6. For a set E in a metric space $(\mathcal{M}, d)$, we defined the interior, E°, to be the set of all interior points of E. Show that E° is the largest open set contained in E.

   Hint: Show that E° is open, $E^\circ \subseteq E$, and if U is open and $U \subseteq E$, then $U \subseteq E^\circ$.

7. Consider the metric space $\mathcal{M} = [1, \infty)$ with $d(x, y) = \left| \dfrac{1}{x} - \dfrac{1}{y} \right|$. Show that $(\mathcal{M}, d)$ is not complete.

   Hint: Show that the sequence $x_n = n$, $n \in \mathbb{N}$, is a Cauchy sequence that does not converge.

8. Prove or disprove: An inner product space is necessarily complete.

9. Prove or disprove: if $(\mathcal{M}, d)$ is a complete metric space and $F \subset \mathcal{M}$ is a closed subset, then (F, d) is a complete metric space.

10. Verify the following inequalities, which relate the various p-norms on $\mathbb{R}^n$,

$$\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 \leq n \|x\|_\infty , \quad \text{and} \quad \|x\|_1 \leq \sqrt{n}\, \|x\|_2 .$$

   Balls in the norm $\|\cdot\|_p$ are often written $B_r^p(x)$. Show that the inequalities above imply that the following sets are *nested:*

$$B^2_{r/n}(x) \subseteq B^\infty_{r/n}(x) \subseteq B^1_r(x) \subseteq B^2_r(x) \subseteq B^\infty_r(x).$$

11. Prove that a norm $\| \cdot \|$ on a real vector space V is induced by an inner product if and only if it satisfies the **parallelogram law**:

$$\|x + y\|^2 + \|x - y\|^2 \;=\; 2\|x\|^2 + 2\|y\|^2, \quad \forall x, y \in V.$$

   Hint: First, show that any inner product on V induces a norm that satisfies the parallelogram law. Second, given a norm $\| \cdot \|$ on V that satisfies the parallelogram law, show that

$$\langle x, y \rangle \;=\; \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 \right) \qquad (\heartsuit)$$

   defines an inner product on V, and that the norm induced by this inner product is $\| \cdot \|$.

12. For each of the real vector spaces $\mathbb{R}^n$, ℓ^p, and $C[a,b]$, prove that the only p-norm that is induced by an inner product is the Euclidean norm.

Hint: Show that a p-norm satisfies the parallelogram law if and only if $p = 2$.

Version of March 24, 2025 @ 11:00