Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 4 (Solutions)

This assignment was **due** on **Tuesday 12 November 2019 at 2:25pm** via crowdmark.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. In each part of this problem, the function f is defined by the formula

$$f(x) = \sqrt{|x|}\,. \qquad (\heartsuit)$$

Pay close attention to the domain of the function in each part and consider the statement

$$\lim_{x\to 2} f(x) = \sqrt{2}\,. \qquad (\spadesuit)$$

Does statement ($\spadesuit$) make sense for the given domain? If not, why not? If statement ($\spadesuit$) does make sense, then either prove or disprove it directly from the ε-δ definition of a limit.

(a) $f : \mathbb{R} \to \mathbb{R}$.

Solution: The statement ($\spadesuit$) makes sense because the function $f(x)$ is well-defined at each $x \in \mathbb{R}$. To prove that the stated limit is, in fact, correct, we must show that $\forall\varepsilon > 0\ \exists\delta > 0$ such that $|x-2| < \delta \implies |f(x) - f(2)| < \varepsilon$. Since $f(2) = \sqrt{2}$, we must show $\left|\sqrt{|x|} - \sqrt{2}\right| < \varepsilon$. To that end, note that

$$\begin{aligned}\left|\sqrt{|x|} - \sqrt{2}\right| &= \left|\sqrt{|x|} - \sqrt{2}\right| \cdot \frac{\left|\sqrt{|x|} + \sqrt{2}\right|}{\left|\sqrt{|x|} + \sqrt{2}\right|} \\ &= \left|\frac{|x| - 2}{\sqrt{|x|} + \sqrt{2}}\right| \le \big||x| - 2\big| \qquad \forall x \in \mathbb{R}.\end{aligned}$$

Note further that if $x \ge 0$ then the absolute value bars around x in the final expression can be dropped to obtain $|x-2|$. Therefore, given $\varepsilon > 0$, choose $\delta = \min\{2, \varepsilon\}$ (we need $\delta \le 2$ to ensure $x \ge 0$). Then $|x-2| < \delta \iff \big||x|-2\big| < \delta \implies \left|\sqrt{|x|} - \sqrt{2}\right| < \varepsilon$, as required. $\square$

An alternative proof that does not require rationalizing the numerator can be constructed as follows. This is definitely not as slick, but it illustrates that one can find quite different proofs for the same result. Note that $\left|\sqrt{|x|} - \sqrt{2}\right| < \varepsilon$ is equivalent to

$$-\varepsilon < \sqrt{|x|} - \sqrt{2} < \varepsilon\,.$$

We are interested in x near 2, so we can restrict attention to $x > 0$ (*i.e.,* we will take $\delta < 2$), and hence what we need to show is that

$$
\begin{aligned}
&-\varepsilon < \sqrt{x} - \sqrt{2} < \varepsilon \\
\iff\ &\sqrt{2} - \varepsilon < \sqrt{x} < \sqrt{2} + \varepsilon\,.
\end{aligned}
$$

We are interested in arbitrarily small ε, so we can restrict attention to $\varepsilon < \sqrt{2}$, in which case we can square the sequence of inequalities above and find that what we need to show is that

$$
\begin{aligned}
&(\sqrt{2} - \varepsilon)^2 < x < (\sqrt{2} + \varepsilon)^2 \\
\iff\ &(\sqrt{2} - \varepsilon)^2 - 2 < x - 2 < (\sqrt{2} + \varepsilon)^2 - 2 \\
\iff\ &\varepsilon^2 - 2\varepsilon\sqrt{2} + 2 - 2 < x - 2 < \varepsilon^2 + 2\varepsilon\sqrt{2} + 2 - 2 \\
\iff\ &\varepsilon^2 - 2\varepsilon\sqrt{2} < x - 2 < \varepsilon^2 + 2\varepsilon\sqrt{2} \\
\Longleftarrow\ &-\varepsilon^2 - 2\varepsilon\sqrt{2} < x - 2 < \varepsilon^2 + 2\varepsilon\sqrt{2} \\
\iff\ &|x - 2| < \varepsilon^2 + 2\varepsilon\sqrt{2} \\
\Longleftarrow\ &|x - 2| < \varepsilon^2\,.
\end{aligned}
$$

Thus, given $0 < \varepsilon < \sqrt{2}$, choose $\delta = \varepsilon^2$. Then from the sequence of steps above, $|x-2| < \delta \implies \left|\sqrt{|x|} - \sqrt{2}\right| < \varepsilon$, as required. □

(b) $f : \mathbb{Q} \to \mathbb{R}$.

Solution: The statement (♠) makes sense because the function $f(x)$ is well-defined at each $x \in \mathbb{Q}$ (and the limit, $\sqrt{2}$, is in the co-domain, $\mathbb{R}$). To prove that the stated limit is, in fact, correct, we must show that $\forall\varepsilon > 0\ \exists\delta > 0$ such that, if $|x - 2| < \delta$ **and** $x \in \mathbb{Q}$, then $|f(x) - f(2)| < \varepsilon$. The argument in part (a) works equally well here, since we have established that we can find a suitable δ without the restriction of the domain to $\mathbb{Q}$, so it certainly applies when we restrict attention to rational points. □

Note: f here is the restriction to $\mathbb{Q}$ of a function that is well-defined and continuous on $\mathbb{R}$. But consider the function

$$
g(x) = \begin{cases} \sqrt{|x|} & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}
$$

g agrees with f when restricted to $\mathbb{Q}$, hence g is continuous as a function on $\mathbb{Q}$. But g is not continuous (except at one point) as a function on $\mathbb{R}$.

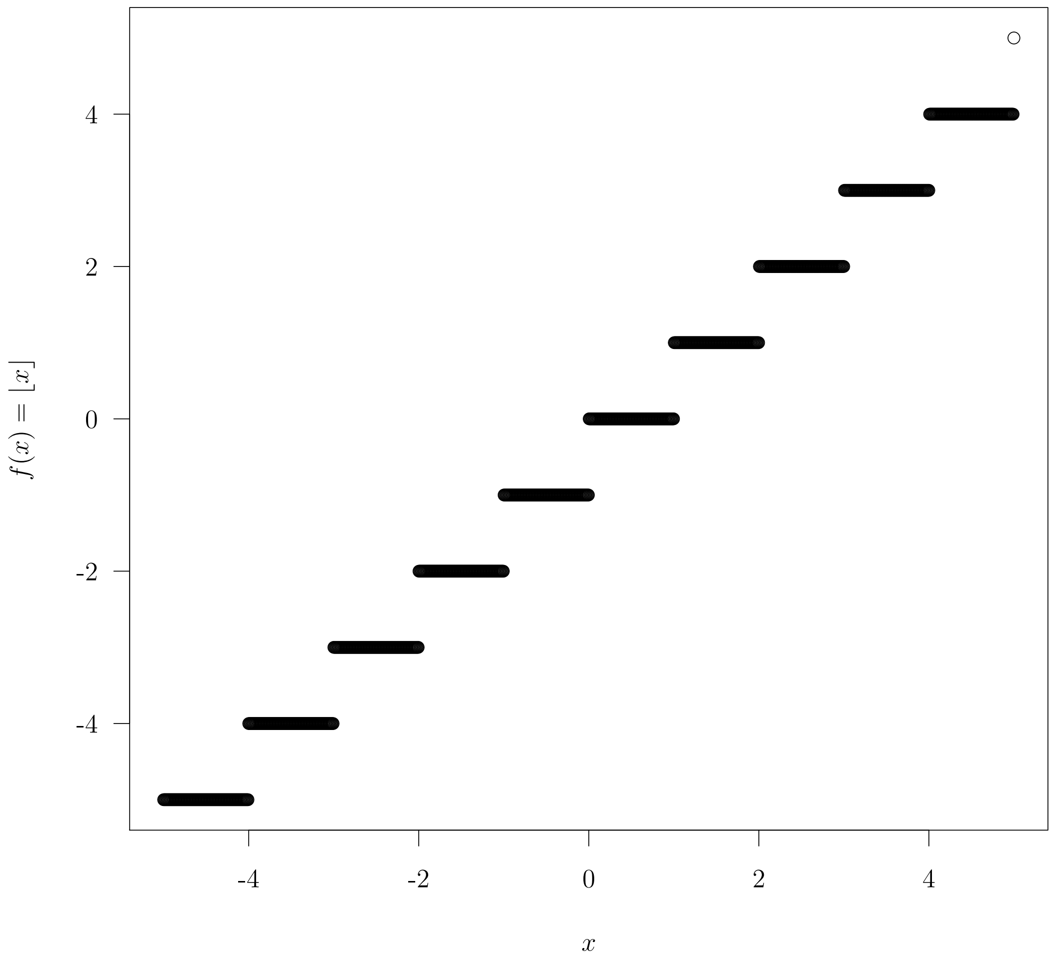
(c) $f : \mathbb{Z} \to \mathbb{R}$.

Solution: The statement (♠) makes sense because the function $f(x)$ is well-defined at each $x \in \mathbb{Z}$ (and the limit, $\sqrt{2}$, is in the co-domain, $\mathbb{R}$). Again, the function f is continuous when restricted to $\mathbb{Z}$, but note that *any* function $f : \mathbb{Z} \to \mathbb{R}$ is continuous because if we choose $\delta = 1/2$, say, then the only point in the domain ($\mathbb{Z}$) that satisfies $|x - 2| < \delta$ is $x = 2$ itself. □

2. The ***floor*** function is defined for all $x \in \mathbb{R}$ by $\lfloor x \rfloor$ = the greatest integer less than or equal to x, *i.e.*, the greatest $n \in \mathbb{Z}$ such that $n \leq x$. Determine the points of continuity of the following functions:

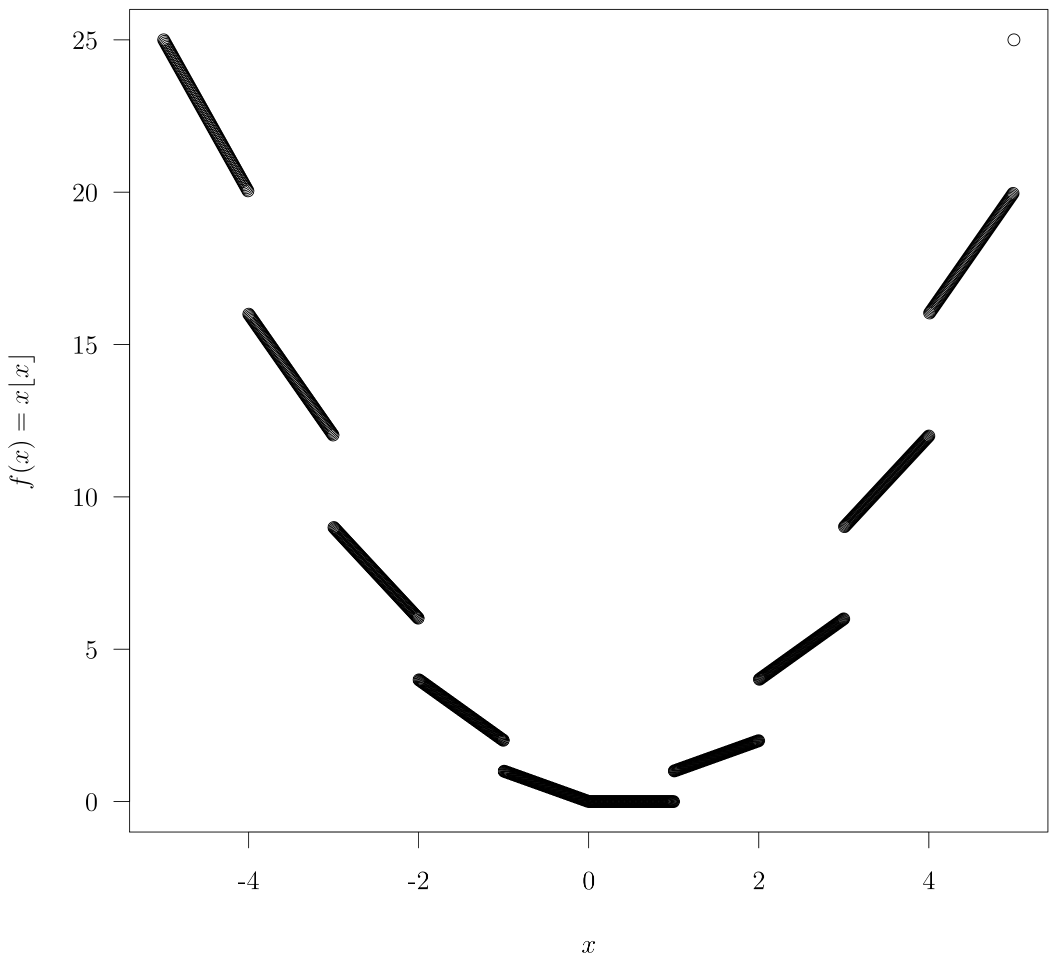
   (a) $f(x) = \lfloor x \rfloor$;

   Solution: f is locally constant (and hence continuous) on each open interval $(n, n+1)$, where $n \in \mathbb{Z}$. f has a jump discontinuity at each $n \in \mathbb{Z}$.



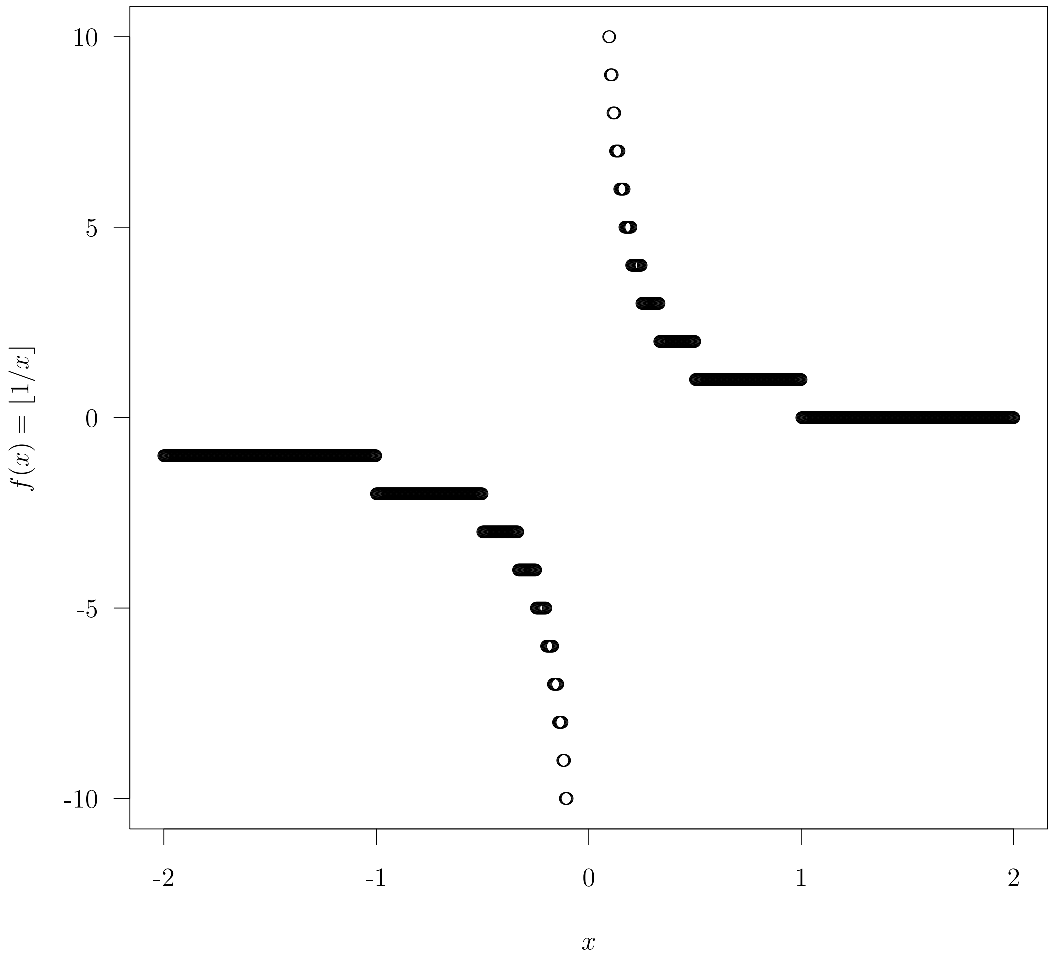
(b) $f(x) = x\lfloor x \rfloor$;

Solution: This function is continuous on each open interval $(n, n+1)$, where $n \in \mathbb{Z}$, and at $x = 0$. f has a jump discontinuity at each $n \in \mathbb{Z} \setminus \{0\}$.



(c) $f(x) = \lfloor 1/x \rfloor$, $x \neq 0$.

Solution: This function is continuous on each open interval $(\frac{1}{n}, \frac{1}{n+1})$, where $n \in \mathbb{Z}^{<-1}$, on each open interval $(\frac{1}{n+1}, \frac{1}{n})$, where $n \in \mathbb{Z}^{>0}$, and throughout the intervals $(-\infty, -1)$ and $(1, \infty)$. f has a jump discontinuity at each reciprical non-zero integer, $x = \frac{1}{n}$, $n \in \mathbb{Z} \setminus \{0\}$.



3. Show that the function $f(x) = 1/x^2$ is (a) uniformly continuous on $[1, \infty)$, but (b) not uniformly continuous on $(0, \infty)$.

Solution: (a) Given $\varepsilon > 0$ we must find $\delta > 0$ such that if $x, y \geq 1$ and $|x - y| < \delta$ then $|(1/x^2) - (1/y^2)| < \varepsilon$. Note that

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{y^2}\right| &= \left|\frac{x^2 - y^2}{x^2y^2}\right| \\
&= |x - y| \left|\frac{x + y}{x^2y^2}\right| \\
&= |x - y| \left|\frac{1}{xy^2} + \frac{1}{x^2y}\right| \\
&\leq |x - y| \cdot 2 \qquad \because x, y \geq 1.
\end{aligned}$$

Therefore, given $\varepsilon > 0$, choose $\delta = \varepsilon/2$. Then the above calculation implies that if $|x - y| < \delta$ then

$$\left|\frac{1}{x^2} - \frac{1}{y^2}\right| \leq 2\,|x - y| < 2\frac{\varepsilon}{2} = \varepsilon,$$

as required. □

(b) Consider $\varepsilon = 1$. Given $\delta > 0$, we must find $x, y \in (0, \infty)$ such that $|x - y| < \delta$, yet $|(1/x^2) - (1/y^2)| \geq 1$. To that end, suppose $y = 2x$ (and $x > 0$). Then

$$|x - y| = |-x| = x,$$

and

$$\begin{aligned}
\left|\frac{1}{x^2} - \frac{1}{y^2}\right| &= \left|\frac{1}{x^2} - \frac{1}{(2x)^2}\right| \\
&= \left|\frac{1}{x^2} - \frac{1}{4x^2}\right| \\
&= \frac{3/4}{x^2} \\
&> \frac{1/2}{x^2}.
\end{aligned}$$

Now note that

$$\begin{aligned}
\frac{1/2}{x^2} > 1 &\iff x^2 < \frac{1}{2} \\
&\iff x < \frac{1}{\sqrt{2}}
\end{aligned}$$

Therefore, given $\delta > 0$, choose $x = \min\{\frac{\delta}{2}, \frac{1}{2\sqrt{2}}\}$ and $y = 2x$. Then, the above calculations show that $|x - y| < \delta$, yet $|(1/x^2) - (1/y^2)| > 1$, as required. □

4. Prove that a continuous function maps closed intervals to closed intervals. Thus, if $a < b$ and $f : [a, b] \to \mathbb{R}$ is continuous then $f([a, b])$ (the range of f) is a closed interval. *Hint:* Intermediate Value Theorem.

   Note: This is a special case of the more general theorem mentioned in class that a continuous function maps compact sets to compact sets.

   Solution: First, since $[a, b]$ is a compact set, f attains a maximum value (say M) and minimum value (say m) on $[a, b]$. Therefore, there exist $x, y \in [a, b]$ such that $f(x) = m$, $f(y) = M$, and $m \leq f(z) \leq M$ for all $z \in [a, b]$. Thus, $f([a, b]) \subseteq [m, M]$. We will show that $[m, M] \subseteq f([a, b])$ to complete the proof. If $m = M$ then f is constant and the range of f is a single point or, equivalently, the degenerate closed interval $[m, m]$. Now suppose $m < M$. Then the Intermediate Value Theorem implies that for every $L \in (m, M)$, there exists $z \in (x, y)$ (if $x < y$) or $z \in (y, x)$ (if $x > y$) such that $f(z) = L$. Thus, $f([a, b])$ contains every point between m and M, *i.e.,* $[m, M] \subseteq f([a, b])$. Therefore, $f([a, b]) = [m, M]$, *i.e.,* the range of f is a closed interval. $\square$

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