Mathematics 3A03 Real Analysis I

Winter 2025 ASSIGNMENT 4

Topic: Function Sequences

Participation deadline: **Wednesday 12 March 2025 at 11:25am**

The meaning of the participation deadline is that you must answer the multiple choice questions on `childsmath` before that deadline in order to receive participation credit for the assignment. The `childsmath` poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on `childsmath`. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on `crowdmark`. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on `crowdmark` for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via `crowdmark`. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Considering $f_n(x) = \sqrt[n]{x}$ on $[0,1]$, which of the following statements are true?

   ☐ $\{f_n\}$ does not converge;

   ☐ $\{f_n\}$ converges pointwise;

   ☐ $\{f_n\}$ converges uniformly;

   If the sequence converges pointwise, determine the pointwise limit on the indicated interval.

2. Considering $f_n(x) = \begin{cases} 0, & x \leq n, \\ x-n & x \geq n, \end{cases}$ on $[a,b]$ and on $\mathbb{R}$, which of the following statements are true?

   ☐ $\{f_n\}$ does not converge;

   ☐ $\{f_n\}$ converges pointwise on $[a,b]$;

   ☐ $\{f_n\}$ converges pointwise on $\mathbb{R}$;

   ☐ $\{f_n\}$ converges uniformly on $[a,b]$;

   ☐ $\{f_n\}$ converges uniformly on $\mathbb{R}$.

   If the sequence converges pointwise, determine the pointwise limit on the indicated interval.

3. Considering $f_n(x) = \dfrac{e^x}{x^n}$, on $(1,\infty)$, which of the following statements are true?

   ☐ $\{f_n\}$ does not converge;

   ☐ $\{f_n\}$ converges pointwise;

   ☐ $\{f_n\}$ converges uniformly;

   If the sequence converges pointwise, determine the pointwise limit on the indicated interval.

4. Consider the series
$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}.$$
   Which of the following statements are true about this series?

   ☐ does not converge for any $x \in \mathbb{R}$;

   ☐ converges pointwise on a non-empty set but not on all of $\mathbb{R}$;

   ☐ converges pointwise on $\mathbb{R}$;

☐ converges uniformly on a non-empty set but not on all of $\mathbb{R}$;

☐ converges uniformly on $\mathbb{R}$.

Additional practice problems

5. Consider the sequence of functions $\{f_n\}$, where

$$f_n(x) = \frac{1}{n(1+nx^2)}\,, \qquad x \in \mathbb{R}.$$

   (a) For which $x \in \mathbb{R}$ does the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converge pointwise?

   (b) For which $a, b \in \mathbb{R}$ $(a < b)$ does the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n$ converge uniformly on $[a, b]$ to a continuous function?

   (c) For which $a, b \in \mathbb{R}$ $(a < b)$ does the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n$ converge uniformly on $[a, b]$ to a differentiable function f? For such a, b, is f' necessarily the uniform limit of $\displaystyle\sum_{n=1}^{\infty} f_n'$?

   (d) Rather than closed, finite intervals $[a, b]$, consider infinite open intervals (a, ∞). Answer parts (b) and (c) again after revising them to read "For which $a \in \mathbb{R}$ does the series… converge uniformly on (a, ∞) to…".

6. Prove or disprove: If $f(x) = \displaystyle\sum_{n=1}^{\infty} a_n x^n$ is an even function, then $a_n = 0$ for n odd, and if f is an odd function, then $a_n = 0$ for n even.

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