Mathematics 3A03 Real Analysis I

Winter 2025 ASSIGNMENT 3

Topic: Topology of $\mathbb{R}$

Participation deadline: **Monday 24 February 2025 at 11:25am**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Let $E = \{x : \sqrt{2} \le x \le \sqrt{3},\ x \notin \mathbb{Q}\}$. Considering E as a subset of $\mathbb{R}$, which of the following statements is true?

   ☐ E is open in $\mathbb{R}$.

   False. Every point in an open set is contained in an open intervals that is a subset of the set. But any open interval containing any point in E contains rational numbers, *i.e.,* points *not* in E. ☐

   ☐ E is closed in $\mathbb{R}$.

   False. $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. In particular, $\frac{3}{2}$ is an accumulation point of E, but $\frac{3}{2} \notin E$. ☐

   ■ E is neither open nor closed in $\mathbb{R}$.

   True, based on the previous two results.

In addition:

   ☐ Find the interior of E in $\mathbb{R}$.

   $E^\circ = \varnothing$. The argument above that E is not open shows that no open interval is a subset of E, hence E has no interior. ☐

   ☐ Find the closure of E in $\mathbb{R}$.

   $\overline{E} = [\sqrt{2}, \sqrt{3}]$.

   For any rational point $q \in [\sqrt{2}, \sqrt{3}]$, there is a sequence of irrational numbers in E that approaches q ($\mathbb{Q}$ is dense in $\mathbb{R}$).

   ☐ Find the boundary of E in $\mathbb{R}$.

   $\partial E = [\sqrt{2}, \sqrt{3}]$. Both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$.

2. Which of the following statements are true for a set $E \subseteq \mathbb{R}$?

   ■ No interior point can be a boundary point;

   True. For any $x \in E^\circ \subseteq \mathbb{R}$, there exists $c > 0$ such that $(x - c, x + c) \subseteq E$. But $(x - c, x + c)$ is therefore a neighbourhood of x that contains no points of $\mathbb{R} \setminus E$, *i.e.,* x is not a boundary point of E. ☐

   ■ it is possible for an accumulation point to be a boundary point;

   True. Suppose $E = (0, 1]$. The point $0 \notin E$, but 0 is an accumulation point of E since any neighbourhood of 0 contains points of E. ☐

   ■ every isolated point must be a boundary point.

   True. Suppose x is an isolated point of a set $E \subset \mathbb{R}$. Then there is a neighbourhood $(x-c, x+c)$ of x for which x is the only element of E. Any other neighbourhood $(x-d, x+d)$ of x contains x, and regardless of whether d is less than or greater than c, there are points of $(x - c, x + c) \setminus \{x\}$ in $(x - d, x + d)$, so x is a boundary point of E. ☐

3. Which of the following statements are true?

■ a set E is closed iff $\overline{E} = E$;

True. For any set E, $\overline{E} = E \cup E'$, where E' is the set of accumulation points of E. By definition, a set is closed iff it contains all its accumulation points, *i.e.*, $E' \subseteq E$. Thus, we must prove that

$$E' \subseteq E \iff E \cup E' = E\,.$$

$(\implies)$ If $A \subseteq B$ then for any other set C, $C \cup A \subseteq C \cup B$. Therefore, $E' \subseteq E \implies E \cup E' \subseteq E \cup E = E$.

$(\impliedby)$ The meaning of $E \cup E' = E$ is that $E \cup E' \subseteq E$ and $E \cup E' \supseteq E$. But $E \cup E' \subseteq E$ implies that $E' \subseteq E$. □

■ a set E is open iff $E^\circ = E$.

A set E is open iff for each point $x \in E$ there is a neighbourhood U of x such that $U \subseteq E$, *i.e.*, iff every point of E is an interior point of E, *i.e.*, iff the set of all interior points of E is entire set E, *i.e.*, iff $E^\circ = E$. □

4. Let $E = [0, 1]$ be the closed unit interval. Which of the following statements are true?

■ E can be expressed as an intersection of a sequence of open sets;

True. For example,

$$[0,1] = \bigcap_{n=1}^{\infty} \Big(-\frac{1}{n}, 1 + \frac{1}{n}\Big)\,.$$

□ E can be expressed as a union of a sequence of open sets;

False. Note that any union of open sets is open, yet $[0, 1]$ is not open so it cannot be expressed as union of open sets. (It is important to state that $[0, 1]$ is "not open". The fact that $[0, 1]$ is closed does not imply on its own that $[0, 1]$ is not open; recall that $\mathbb{R}$ is both open and closed.)

To prove that the union of a sequence of open sets is open, let

$$\mathcal{U} = \bigcup_{n=1}^{\infty} U_n$$

be the union of a sequence of open sets $\{U_n\}$. If $x \in \mathcal{U}$ then there must be an open set U_i such that $x \in U_i$. But $U_i \subseteq \mathcal{U}$, which implies x is an interior point of $\mathcal{U}$. Since x was an arbitrary point of $\mathcal{U}$, it follows that all points of $\mathcal{U}$ are interior points, *i.e.*, $\mathcal{U}$ is open. □

□ E can be expressed as a union of uncountably many open sets.

Also false. To prove more generally (and slightly more abstractly) that *any* union of open sets is open, suppose $\mathcal{U}$ is a union of open sets. If $x \in \mathcal{U}$ then there must be an open set $U \subseteq \mathcal{U}$ such that $x \in U$, which implies x is an interior point of $\mathcal{U}$. As above, it follows that $\mathcal{U}$ is open. □

Additional practice problems

5. Determine which of the following sets are open, which are closed, and which are neither open nor closed.

(a) $(-\infty, 0) \cup (0, \infty)$

Open, not closed. It is a union of open intervals, hence open. The origin is an accumulation point that is not in the set, so it is not closed.

(b) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

Not open, not closed. It contains no intervals so can't be open. It does not contain its accumulation point at 0, so it is not closed.

(c) $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

Not open, closed. The missing accumulation point is now included.

(d) $(0,1) \cup (1,2) \cup (2,3) \cup (3,4) \cup \cdots \cup (n, n+1) \cup \cdots$

Open, not closed. It is a union of open intervals, hence open. The set does not contain the accumulation points at the non-negative integers.

(e) $(\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{8}, \frac{1}{4}) \cup (\frac{1}{16}, \frac{1}{8}) \cup \cdots$

Open, not closed. It is a union of open intervals, hence open. The set does not contain the accumulation points at $1/n$ for each $n \in \mathbb{N}$ (nor does it contain the accumulation point at 0).

(f) $\{x : |x - \pi| < 1\}$

Open, not closed. This is the open interval $(\pi - 1, \pi + 1)$.

(g) $\{x : x^2 < 2\}$

Open, not closed. This is the open interval $(-\sqrt{2}, \sqrt{2})$.

(h) $\mathbb{R} \setminus \mathbb{N}$

Open, not closed. The complement $\mathbb{N}$ is closed, hence this set is open. Each point in $\mathbb{N}$ is an accumulation point of the set, but is not in the set, so the set is not closed.

(i) $\mathbb{R} \setminus \mathbb{Q}$

Not open, not closed. Any open interval containing an irrational number also contains a rational number, so the set is not open. Every point in $\mathbb{Q}$ is an accumulation point of $\mathbb{Q}^{\mathrm{c}}$ so $\mathbb{Q}^{\mathrm{c}}$ is not closed.

6. Prove or disprove: If $E \subseteq \mathbb{R}$ and E is both open and closed then $E = \mathbb{R}$ or $E = \varnothing$.

The claim is true.

As discussed in class, both $\mathbb{R}$ and $\varnothing$ are both open and closed. Suppose $E \neq \varnothing$ and E is both open and closed. We will show that $E = \mathbb{R}$.

Since E is non-empty, it contains at least one point, say x. Since E is open, there is a neighbourhood of x that is contained in E. Note that any interval U containing x can be written as the union of two half-open intervals, $U = (x - \ell, x] \cup [x, x + r)$, where $\ell, r > 0$. Let

$$R = \sup\left\{r \in \mathbb{R} : [x, x+r) \subseteq E\right\}, \tag{$*$}$$

where we will use the notation $R = \infty$ if the least upper bound does not exist. If $R < \infty$ (*i.e.*, $R \in \mathbb{R}$) then—since E is closed—we must have $[x, x+R] = \overline{[x, x+R)} \subseteq E$. But then—since $x + R \in E$ and E is open—there is a neighbourhood of $x + R$ that is contained in E, contradicting R being the least upper bound in (*). Therefore, $R = \infty$. Now let

$$L = \inf\left\{\ell \in \mathbb{R} : (x - \ell, x] \subseteq E\right\}. \tag{$**$}$$

Then, by a similar argument we must have $L = -\infty$. Thus, $(-\infty, \infty) \subseteq E$, *i.e.*, $E = \mathbb{R}$. $\square$

7. Prove directly (*i.e.,* from the definition of the Bolzano-Weierstrass property) that

   (a) the interval $[0, \infty)$ does not have the Bolzano-Weierstrass property;

   We will demonstrate that there is a sequence of non-negative real numbers that has no convergent subsequence. Consider the sequence $\{a_n\}$, where $a_n = n$ for all n, *i.e.,* $\{a_n\}$ is the sequence of natural numbers. $\{a_n\} \subset [0, \infty)$ and $a_n \to \infty$ as $n \to \infty$. Moreover, any subsequence of $\{a_n\}$ also diverges to ∞, since the only bounded subsets of $\mathbb{N}$ are finite. □

   (b) the union of two sets that have the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.

   Let $F = F_1 \cup F_2$, where F_1 and F_2 are sets with the Bolzano-Weierstrass property. Thus, for $i = 1$ or 2, any sequence in F_i contains a subsequence that converges to a point in F_i. Let $\{s_n\}$ be a sequence in F. The sequence $\{s_n\}$ must contain infinitely many terms in at least one of F_1 or F_2 (if not then there would be only finitely many points in the sequence), so assume wlog that $\{s_n\}$ contains infinitely many points from F_1. Let $\{t_n\}$ be the subsequence of $\{s_n\}$ that contains only the points of $\{s_n\}$ that are in F_1. This is an infinite sequence in F_1 so—since F_1 has the Bolzano-Weierstrass property—$\{t_n\}$ contains a subsequence that converges to a point, say L, in F_1. But that subsequence of $\{t_n\}$ that converges to a point in F_1 is also a subsequence of the original sequence $\{s_n\}$ that converges to a point in F, as required. □

8. Let $E = \left\{x \in \mathbb{Q} \mid -\sqrt{2} < x < 0\right\}$.

   (a) Find the closure of E in $\mathbb{R}$.

   $\overline{E} = [-\sqrt{2}, 0]$. To see this, note that the sequence $\{-\frac{1}{n} : n \in \mathbb{N}\}$ is a sequence in E that converges to 0, hence $0 \in \overline{E}$. In addition, since $E = [-\sqrt{2}, 0) \cap \mathbb{Q}$, and $\mathbb{Q}$ is dense in $\mathbb{R}$, for any $x \in [-\sqrt{2}, 0)$ there is a sequence $\{q_n\} \subset [-\sqrt{2}, 0) \cap \mathbb{Q}$ such that $q_n \to x$, hence $[-\sqrt{2}, 0) \subset \overline{E}$. Thus $[-\sqrt{2}, 0] \subseteq \overline{E}$. Suppose there is another point in $\overline{E}$, say $x > 0$. Then x is isolated from $[-\sqrt{2}, 0]$, since we can find a neighbourhood of x that does not intersect $[-\sqrt{2}, 0]$ (*e.g.,* the interval $(\frac{x}{2}, \frac{3x}{2})$); so x cannot, in fact, be in $\overline{E}$ (and similarly for $x < -\sqrt{2}$). Thus, $\overline{E} = [-\sqrt{2}, 0]$. □

   (b) Is E closed?

   No, since $E \neq \overline{E}$. □

   (c) Find the interior of E in $\mathbb{R}$.

   $E^\circ = \varnothing$. To see this, suppose $E^\circ \neq \varnothing$. Then there exists $x \in E$ and $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset E$. The irrational numbers $\mathbb{Q}^c$ are dense in $\mathbb{R}$, so there is some irrational $y \in (x-\varepsilon, x+\varepsilon)$. Hence $y \in E \subset \mathbb{Q}$. $\Rightarrow\Leftarrow$ □

   (d) Is E open?

   No, since $E^\circ \neq E$. □

   (e) (Bolzano-Weierstrass Property) Does every sequence of points in E have a subsequence that converges to a point in E? If so, prove it. Otherwise, construct a sequence with no subsequence converging in E.

   No. In part (a) we showed there is a sequence in E that converges to 0, so every subsequence of this sequence also converges to 0. But $0 \notin E$, so we have a sequence in E that does not have a subsequence that converges to a point in E. □

(f) (Heine-Borel Property) Does every open cover of E have a finite subcover? If so, prove it. Otherwise, construct an open cover that has no finite subcover.

No. Consider the open intervals $U_n = (-\sqrt{2}, -\frac{1}{n})$ for each $n \in \mathbb{N}$. Then the collection $\mathcal{U} = \{U_n\}$ is an open over of E since $\bigcup_{i=1}^{n} U_n = E$. But no finite subcollection of $\mathcal{U}$ covers E, so there is no finite subcover. □

9. Prove that the interval $[0, 1]$ is compact, directly from the definitions of each of the three equivalent characterizations of compactness:

(a) $[0, 1]$ is closed and bounded;

If $x \in [0, 1]$ then $0 \leq x \leq 1$, hence $[0, 1]$ is bounded. Suppose $[0, 1]$ is not closed. Then $[0, 1]$ has an accumulation point $x \notin [0, 1]$. Hence either $x < 0$ or $x > 1$. Suppose $x < 0$. Then let $\delta = -\frac{x}{2}$ and observe that $(x - \delta, x + \delta)$ is a neighbourhood of x that contains no points of $[0, 1]$ other than x itself. Hence x is not an accumulation point of $[0, 1]$. $\Rightarrow\Leftarrow$. Therefore, $[0, 1]$ is closed. □

(b) $[0, 1]$ has the Bolzano-Weierstrass property;

This is a special case of the theorem proved in class that any closed and bounded set has the Bolzano-Weierstrass property. The proof is identical, so check the slides.

(c) $[0, 1]$ has the Heine-Borel property.

This requires a clever argument, which is a key piece of the proof of the general Heine-Borel theorem. Suppose, in order to derive a contradiction, that $[0, 1]$ does not have the Heine-Borel property, *i.e.,* there is an open cover $\mathcal{U}$ of $[0, 1]$ that contains no finite subcover. Thus, infinitely many sets in $\mathcal{U}$ are required to cover $[0, 1]$. Consider the two closed subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, obtained by bisecting $[0, 1]$. It must be that at least one of these subintervals cannot be covered by finitely many sets in $\mathcal{U}$. Call this subinterval I_1 (if neither of the two subintervals can be covered by finitely many sets in $\mathcal{U}$ then it doesn't matter which one we choose). Now, by a similar argument, we can bisect I_1 and find that one of its two subintervals (call it I_2) cannot be covered by finitely many sets in $\mathcal{U}$. Continuing inductively, we have a nested sequence of closed intervals

$$[0, 1] \equiv I_0 \supset I_1 \supset I_2 \supset \cdots,$$

none of which can be covered by finitely many sets in $\mathcal{U}$. Note that the length of I_n is $\frac{1}{2^n}$. Now choose a sequence $\{x_n\}$ where $x_k \in I_k$ for each k. Because the intervals are nested and shrink in length to zero, $\{x_n\}$ is a Cauchy sequence of real numbers, and therefore converges, say $x_n \to L$. Moreover, we must have $L \in I_k$ for all $k \in \mathbb{N}$ (otherwise we could isolate L from I_k for all k large enough, contradicting L being the limit of the sequence). Finally, since $L \in [0, 1]$, there exists $U \in \mathcal{U}$ such that $L \in U$. But for sufficiently large n, we must have $I_n \subset U$, which means I_n is covered by the single set $U \in \mathcal{U}$, contradicting the conclusion above that no I_k can be covered by finitely many sets in $\mathcal{U}$. $\Rightarrow\Leftarrow$ Thus, $[0, 1]$ must have the Heine-Borel property. □