Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 3 (Solutions)

This assignment was **due** on **Tuesday 22 October 2019 at 2:25pm** via crowdmark.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. Consider the sequence $\{a_n\}$ defined by

$$a_1 = 0.1,\ a_2 = 0.12,\ a_3 = 0.123,\ \ldots,\ a_{12} = 0.123456789101112,\ \ldots$$

Prove that $\{a_n\}$ converges.

Solution: $\{a_n\}$ is bounded ($0 < a_n < 0.2$ for all n) and increasing, hence by the Monotone Convergence Theorem, $\{a_n\}$ converges. □

2. Suppose $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences and let $c_n = |a_n - b_n|$ for all n. Prove that $\{c_n\}$ is Cauchy.

Solution:

Since $\{a_n\}$ and $\{b_n\}$ are Cauchy, we know that given any $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_n - a_m| < \frac{\varepsilon}{2}$ and $|b_n - b_m| < \frac{\varepsilon}{2}$. But then $\forall m, n \geq N$ we have

$$\begin{aligned} |c_n - c_m| &= \big||a_n - b_n| - |a_m - b_m|\big| \\ &\leq |(a_n - b_n) - (a_m - b_m)| \\ &= |(a_n - a_m) - (b_n - b_m)| \\ &\leq |a_n - a_m| + |b_n - b_m| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,, \end{aligned}$$

so c_n is Cauchy. □

3. Suppose $\{a_n\}$ is a sequence of real numbers. The following statement looks similar to the Cauchy criterion:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_{n+1} - a_n| < \varepsilon.$$

Prove that there is a sequence $\{a_n\}$ that satisfies this criterion and yet is not Cauchy.

Solution: Let $a_n = \sqrt{n}$, which diverges and is therefore not Cauchy. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned} |a_{n+1} - a_n| &= \sqrt{n+1} - \sqrt{n} \\ &= \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &< \frac{1}{\sqrt{n}}\,. \end{aligned}$$

Therefore, given $\varepsilon > 0$, choose $N = \lceil 1/\varepsilon^2 \rceil + 1$, so $N > \frac{1}{\varepsilon^2}$ and hence $\frac{1}{\sqrt{N}} < \varepsilon$. Then, for all $n \geq N$, we have

$$|a_{n+1} - a_n| < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon\,,$$

as required. □

4. Give examples of functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

   (a) f is one-to-one but not onto;
   Solution: $f(n) = 2n$.

   (b) f is onto but not one-to-one;
   Solution:

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

   (c) f is a bijection that is not the identity.
   Solution: $f(n) = n + 1$.

5. Prove or disprove: There exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that

   (a) f is one-to-one but not onto, g is onto but not one-to-one, and $f \circ g$ is a bijection;
   Solution: If f is not onto, then regardless of the range of g, $f \circ g$ cannot be onto, so $f \circ g$ cannot be a bijection. □

   (b) f is onto but not one-to-one, g is one-to-one but not onto, and $f \circ g$ is a bijection.
   Solution: This can be done by squashing $\mathbb{R}$ to the interval $(-1, 1)$ and then stretching it back out to $(-\infty, \infty)$. For example, let

$$g(x) = \frac{x}{1 + |x|}$$
$$f(x) \begin{cases} \dfrac{x}{1 - |x|} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

   Then g is one-to-one but not onto $\mathbb{R}$, f is onto $\mathbb{R}$ but not one-to-one, and $f(g(x)) = x$ is a bijection of $\mathbb{R}$. □

6. Let U be an uncountable subset of $\mathbb{R}$, and let $U_n = U \cap [-n, n]$ for each $n \in \mathbb{N}$.

   (a) Prove that for some $k \in \mathbb{N}$, U_k is uncountable.
   Solution: Suppose U_k is countable for all $k \in \mathbb{N}$. Then $U = \bigcup_{k \in \mathbb{N}} U_k$ is a countable union of countable sets, and hence is countable. $\Rightarrow\Leftarrow$ □

   (b) Prove that there is a convergent sequence $\{a_n\}$ such that $a_n \in U$ for all n and $a_n \neq a_m$ whenever $n \neq m$.
   Solution: Exploiting part (a), choose $k \in \mathbb{N}$ such that $U_k = U \cap [-k, k]$ is uncountable. In particular, U_k contains countably many distinct points, so there

is a sequence $\{x_n\}$ of distinct points in U_k. But $-k \leq x \leq k$ for all $x \in U_k$, so $-k \leq x_n \leq k$ for all $n \in \mathbb{N}$; thus $\{x_n\}$ is a bounded sequence. Therefore, the Bolzano-Weierstrass theorem implies that $\{x_n\}$ contains a convergent subsequence, say $\{a_n\}$. Moreover, since $\{x_n\}$ is a sequence of distinct points, the subsequence $\{a_n\}$ must also be a sequence of distinct points. ☐

7. Let $E = \{x : \sqrt{2} \leq x \leq \sqrt{3},\ x \notin \mathbb{Q}\}$.

   (a) Prove or disprove: E is open in $\mathbb{R}$.

   Solution: False. Every point in an open set is contained in an open intervals that is a subset of the set. But any open interval containing any point in E contains rational numbers, *i.e.,* points *not* in E. ☐

   (b) Prove or disprove: E is closed in $\mathbb{R}$.

   Solution: False. $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. In particular, $\frac{3}{2}$ is an accumulation point of E, but $\frac{3}{2} \notin E$. ☐

   (c) Find the interior of E in $\mathbb{R}$.

   Solution: $E^\circ = \varnothing$. This follows from the argument in part (a). ☐

   (d) Find the closure of E in $\mathbb{R}$.

   Solution: $\overline{E} = [\sqrt{2}, \sqrt{3}]$.

   (e) Find the boundary of E in $\mathbb{R}$.

   Solution: $\partial E = [\sqrt{2}, \sqrt{3}]$.

8. Prove that the interval $[0,1]$ is compact, directly from the definitions of the each of the three equivalent characterizations of compactness:

   (a) $[0,1]$ is closed and bounded;

   Solution: If $x \in [0,1]$ then $0 \leq x \leq 1$, hence $[0,1]$ is bounded. Suppose $[0,1]$ is not closed. Then $[0,1]$ has an accumulation point $x \notin [0,1]$. Hence either $x < 0$ or $x > 1$. Suppose $x < 0$. Then let $\delta = -x/2$ and observe that $(x-\delta, x+\delta)$ is a neighbourhood of x that contains no points of $[0,1]$ other than x itself. Hence x is not an accumulation point of $[0,1]$. $\Rightarrow\Leftarrow$. Therefore, $[0,1]$ is closed. ☐

   (b) $[0,1]$ has the Bolzano-Weierstrass property;

   Solution: This is a special case of the theorem proved in class that any closed and bounded set has the Bolzano-Weierstrass property.

   (c) $[0,1]$ has the Heine-Borel property.

   Solution: This requires a clever argument, which is a key piece of the proof of the general Heine-Borel theorem. Suppose, in order to derive a contradiction, that $[0,1]$ does not have the Heine-Borel property, *i.e.,* there is an open cover $\mathcal{U}$ of $[0,1]$ that contains no finite subcover. Thus, infinitely many sets in $\mathcal{U}$ are required to cover $[0,1]$. Consider the two closed subintervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, obtained by bisecting $[0,1]$. It must be that at least one of these subintervals cannot be covered by finitely many sets in $\mathcal{U}$. Call this subinterval I_1 (if neither of the two subintervals can be covered by finitely many sets in $\mathcal{U}$ then it doesn't matter which one we choose). Now, by a similar argument, we can bisect I_1 and find that one of its two subintervals (call it I_2) cannot be covered by finitely many sets in $\mathcal{U}$. Continuing inductively, we have a nested sequence of closed intervals

$$[0,1] \equiv I_0 \supset I_1 \supset I_2 \supset \cdots ,$$

none of which can be covered by finitely many sets in $\mathcal{U}$. Note that the length of I_n is $\frac{1}{2^n}$. Now choose a sequence $\{x_n\}$ where $x_k \in I_k$ for each k. Because the intervals are nested and shrink in length to zero, $\{x_n\}$ is a Cauchy sequence, and therefore converges, say $x_n \to L$. Moreover, we must have $L \in I_k$ for all $k \in \mathbb{N}$ (otherwise we could isolate L from I_k for all k large enough, contradicting L being the limit of the sequence). Finally, since $L \in [0,1]$, there exists $U \in \mathcal{U}$ such that $L \in U$. But for sufficiently large n, we must have $I_n \subset U$, which means I_n is covered by just one set in $\mathcal{U}$. $\Rightarrow\Leftarrow$ Thus, $[0,1]$ must have the Heine-Borel property. $\square$