**Mathematics 3A03 Real Analysis I**
**Fall 2019 ASSIGNMENT 3**

This assignment is **due** on **Tuesday 22 October 2019 at 2:25pm**.
**PLEASE NOTE** that you must **submit online** via crowdmark.
You will receive an e-mail from crowdmark with the required link.
Do **NOT** submit a hardcopy of this assignment.

*Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.*

1. Consider the sequence $\{a_n\}$ defined by

$$a_1 = 0.1,\ a_2 = 0.12,\ a_3 = 0.123,\ \ldots,\ a_{12} = 0.123456789101112,\ \ldots$$

   Prove that $\{a_n\}$ converges.

2. Suppose $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences and let $c_n = |a_n - b_n|$ for all $n$. Prove that $\{c_n\}$ is Cauchy.

3. Suppose $\{a_n\}$ is a sequence of real numbers. The following statement looks similar to the Cauchy criterion:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_{n+1} - a_n| < \varepsilon.$$

   Prove that there is a sequence $\{a_n\}$ that satisfies this criterion and yet is not Cauchy.

4. Give examples of functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
   (a) $f$ is one-to-one but not onto;
   (b) $f$ is onto but not one-to-one;
   (c) $f$ is a bijection that is not the identity.

5. Prove or disprove: There exist functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that
   (a) $f$ is one-to-one but not onto, $g$ is onto but not one-to-one, and $f \circ g$ is a bijection;
   (b) $f$ is onto but not one-to-one, $g$ is one-to-one but not onto, and $f \circ g$ is a bijection.

6. Let $U$ be an uncountable subset of $\mathbb{R}$, and let $U_n = U \cap [-n, n]$ for each $n \in \mathbb{N}$.
   (a) Prove that for some $k \in \mathbb{N}$, $U_k$ is uncountable.
   (b) Prove that there is a convergent sequence $\{a_n\}$ such that $a_n \in U$ for all $n$ and $a_n \neq a_m$ whenever $n \neq m$.

7. Let $E = \{x : \sqrt{2} \leq x \leq \sqrt{3},\ x \notin \mathbb{Q}\}$.

   (a) Prove or disprove: $E$ is open in $\mathbb{R}$.

   (b) Prove or disprove: $E$ is closed in $\mathbb{R}$.

   (c) Find the interior of $E$ in $\mathbb{R}$.

   (d) Find the closure of $E$ in $\mathbb{R}$.

   (e) Find the boundary of $E$ in $\mathbb{R}$.

8. Prove that the interval $[0, 1]$ is compact, directly from the definitions of the each of the three equivalent characterizations of compactness:

   (a) $[0, 1]$ is closed and bounded;

   (b) $[0, 1]$ has the Bolzano-Weierstrass property;

   (c) $[0, 1]$ has the Heine-Borel property.