Mathematics 3A03 Real Analysis I
Winter 2025 ASSIGNMENT 3
Topic: Topology of $\mathbb{R}$
Participation deadline: **See course website**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Let $E = \{x : \sqrt{2} \leq x \leq \sqrt{3},\, x \notin \mathbb{Q}\}$. Considering E as a subset of $\mathbb{R}$, which of the following statements is true?

   ☐ E is open in $\mathbb{R}$.

   ☐ E is closed in $\mathbb{R}$.

   ☐ E is neither open nor closed in $\mathbb{R}$.

   In addition:

   ☐ Find the interior of E in $\mathbb{R}$.

   ☐ Find the closure of E in $\mathbb{R}$.

   ☐ Find the boundary of E in $\mathbb{R}$.

2. Which of the following statements are true for a set $E \subseteq \mathbb{R}$?

   ☐ No interior point can be a boundary point;

   ☐ it is possible for an accumulation point to be a boundary point;

   ☐ every isolated point must be a boundary point.

3. Which of the following statements are true?

   ☐ a set E is closed iff $\overline{E} = E$;

   ☐ a set E is open iff $E^\circ = E$.

4. Let $E = [0, 1]$ be the closed unit interval. Which of the following statements are true?

   ☐ E can be expressed as an intersection of a sequence of open sets;

   ☐ E can be expressed as a union of a sequence of open sets;

   ☐ E can be expressed as a union of uncountably many open sets.

Additional practice problems

5. Determine which of the following sets are open, which are closed, and which are neither open nor closed.

   (a) $(-\infty, 0) \cup (0, \infty)$

   (b) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

   (c) $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$

(d) $(0,1) \cup (1,2) \cup (2,3) \cup (3,4) \cup \cdots \cup (n, n+1) \cup \cdots$

(e) $(\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{8}, \frac{1}{4}) \cup (\frac{1}{16}, \frac{1}{8}) \cup \cdots$

(f) $\{x : |x - \pi| < 1\}$

(g) $\{x : x^2 < 2\}$

(h) $\mathbb{R} \setminus \mathbb{N}$

(i) $\mathbb{R} \setminus \mathbb{Q}$

6. Prove or disprove: If $E \subseteq \mathbb{R}$ and E is both open and closed then $E = \mathbb{R}$ or $E = \varnothing$.

7. Prove directly (*i.e.,* from the definition of the Bolzano-Weierstrass property) that

   (a) the interval $[0, \infty)$ does not have the Bolzano-Weierstrass property;

   (b) the union of two sets that have the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.

8. Let $E = \left\{x \in \mathbb{Q} \mid -\sqrt{2} < x < 0\right\}$.

   (a) Find the closure of E in $\mathbb{R}$.

   (b) Is E closed?

   (c) Find the interior of E in $\mathbb{R}$.

   (d) Is E open?

   (e) (Bolzano-Weierstrass Property) Does every sequence of points in E have a subsequence that converges to a point in E? If so, prove it. Otherwise, construct a sequence with no subsequence converging in E.

   (f) (Heine-Borel Property) Does every open cover of E have a finite subcover? If so, prove it. Otherwise, construct an open cover that has no finite subcover.

9. Prove that the interval $[0, 1]$ is compact, directly from the definitions of each of the three equivalent characterizations of compactness:

   (a) $[0, 1]$ is closed and bounded;

   (b) $[0, 1]$ has the Bolzano-Weierstrass property;

   (c) $[0, 1]$ has the Heine-Borel property.