Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 2 (Solutions)

This assignment was **due** on **Tuesday 1 October 2019 at 2:25pm** via crowdmark.

1. Use the formal definition of a limit of a sequence to prove that

   (a) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$;

   Solution: Given $\varepsilon > 0$, we need to find $N \in \mathbb{N}$ such that $\forall n \geq N$, $|1/\sqrt{n}| < \varepsilon$. In order to figure out how to choose N, suppose $|1/\sqrt{n}| < \varepsilon$. Then $1/n < \varepsilon^2$, *i.e.*, $n > 1/\varepsilon^2$. Therefore, given $\varepsilon > 0$, choose $N = \lceil 1/\varepsilon^2 \rceil + 1$. Then

$$\begin{aligned} N &> \left\lceil \frac{1}{\varepsilon^2} \right\rceil \geq \frac{1}{\varepsilon^2} \\ \implies \sqrt{N} &> \frac{1}{\varepsilon} \\ \implies \frac{1}{\sqrt{N}} &< \varepsilon \\ \implies \frac{1}{\sqrt{n}} &< \varepsilon \qquad \forall n \geq N \\ \implies \left| \frac{1}{\sqrt{n}} \right| &< \varepsilon \qquad \forall n \geq N, \end{aligned}$$

   as required. □

   (b) $\lim_{n\to\infty} \frac{n^n - 1}{n^n + 1} = 1.$

   Solution: Given $\varepsilon > 0$ we must show $\exists N \in \mathbb{N}$ such that

$$\left| \frac{n^n - 1}{n^n + 1} - 1 \right| < \varepsilon \qquad \forall n \geq N. \tag{$*$}$$

   In order to determine how to choose N, consider that $\forall n \in \mathbb{N}$

$$\begin{aligned} \left| \frac{n^n - 1}{n^n + 1} - 1 \right| &= \left| \frac{(n^n - 1) - (n^n + 1)}{n^n + 1} \right| = \left| -\frac{2}{n^n + 1} \right| \\ &= \frac{2}{n^n + 1} < \frac{2}{n + 1} < \frac{2}{n}\,. \end{aligned} \tag{$\heartsuit$}$$

   Moreover, if $N \in \mathbb{N}$ then $\forall n \geq N$ we have

$$\frac{2}{n} \leq \frac{2}{N}. \tag{$\spadesuit$}$$

   Consequently, to ensure that (*) holds, it is sufficient to have $2/N < \varepsilon$, *i.e.*, $N > 2/\varepsilon$.

Therefore, given $\varepsilon > 0$, choose $N = \lceil 2/\varepsilon \rceil + 1$. Then, using ($\heartsuit$) and ($\spadesuit$), we have, for all $n \geq N$,

$$\left| \frac{n^n - 1}{n^n + 1} - 1 \right| < \frac{2}{n} < \frac{2}{N} < \varepsilon \,,$$

as required. □

2. Use the formal definition to prove that the following sequences $\{a_n\}$ <u>*diverge*</u> as $n \to \infty$.

   (a) $a_n = \sqrt{n}$;

   Solution: Suppose, in order to derive a contradiction, that $\sqrt{n} \to L$ for some $L \in \mathbb{R}$. Then, given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\forall n \geq N$,

   $$\left|\sqrt{n} - L\right| < \varepsilon \implies -\varepsilon < \sqrt{n} - L < \varepsilon \implies L - \varepsilon < \sqrt{n} < L + \varepsilon \qquad (**)$$

   If $L \leq -\varepsilon$ then $L + \varepsilon \leq 0$ and we have a contradiction (because the right hand inequality cannot be satisfied for <u>any</u> $n \in \mathbb{N}$). So it must be that $L > -\varepsilon$, *i.e.*, $L + \varepsilon > 0$. Now, since (**) holds $\forall n \geq N$, it must hold in particular for $n = \max(N, (\lceil L + \varepsilon \rceil + 1)^2)$. But then $\sqrt{n} \geq \lceil L + \varepsilon \rceil + 1 > L + \varepsilon$, contradicting (**). $\Rightarrow\Leftarrow$ □

   Note: The intention was to work directly from the definition of divergence, but another (simpler) proof could exploit the theorem that convergent sequences are necessarily bounded.

   (b) $a_n = n^{1/k}$ (for fixed $k \in \mathbb{N}$).

   Solution: Suppose, in order to derive a contradiction, that $n^{1/k} \to L$ for some $L \in \mathbb{R}$. Then, given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that $\forall n \geq N$,

   $$\left|n^{1/k} - L\right| < \varepsilon \implies -\varepsilon < n^{1/k} - L < \varepsilon \implies L - \varepsilon < n^{1/k} < L + \varepsilon \,.$$

   Now, as in part (a), we must have $L + \varepsilon > 0$. Then $\forall n \geq N$, we have $n < (L+\varepsilon)^k$. Thus, the set of natural numbers $\{N, N+1, N+2, \ldots\}$ is bounded above by $(L+\varepsilon)^k$, contradicting the Archimedean property. $\Rightarrow\Leftarrow$ □

   Note: We could have used the Archimedean property in part (a), or a variant of the argument in part (a) here. I have purposely given slightly different arguments in order to emphasize that distinct arguments can be used to prove a given proposition.

3. (a) Prove that $\lim\limits_{n\to\infty} a_n = 0$ if and only if $\lim\limits_{n\to\infty} |a_n| = 0$.

   Solution: Suppose $a_n \to L$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - L| < \varepsilon$. But for any $x, y \in \mathbb{R}$, we know from the triangle inequality—as proved in question 3c of Assignment 1—that $||x| - |y|| \leq |x - y|$. Therefore, $\forall n \geq N$, $||a_n| - |L|| \leq |a_n - L| < \varepsilon$. Thus, $|a_n| \to |L|$, as required. □

   (b) Give an example to show that convergence of $\{|a_n|\}$ need not imply convergence of $\{a_n\}$.

   Solution: Let $\{a_n\} = (-1)^n$. Then $\{|a_n|\} = 1$ for all n and hence converges, yet $\{a_n\}$ itself diverges. □

4. Suppose $\lim_{n\to\infty} a_n = a$ and $a > 0$. Prove that

(a) $\exists N \in \mathbb{N}$ such that $a_n > 0$, $\forall n \geq N$;

Solution: Given any $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon$. In particular, consider $\varepsilon = a/2$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|a_n - a| < \frac{a}{2} \implies -\frac{a}{2} < a_n - a < \frac{a}{2} \implies \frac{a}{2} < a_n < \frac{3a}{2}.$$

But $a > 0$, so it follows that $a_n > 0$ $\forall n \geq N$. $\square$

(b) $\exists N' \in \mathbb{N}$ such that $\frac{1}{2}a < a_n < 2a$, $\forall n \geq N'$.

Solution: Since $\dfrac{3a}{2} < 2a$ for any $a > 0$, this follows immediately from part (a). Just take $N' = N$. $\square$