**Mathematics 3A03 Real Analysis I**
**Winter 2025 ASSIGNMENT 2**
**Topic: The (Riemann/Darboux) Integral**
Participation deadline: **See course website**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is integrable on the closed interval $[a, b]$. Then:

   - ● $f$ is necessarily integrable on any closed subinterval of $[a, b]$;
   - ○ There might exist a closed subinterval of $[a, b]$ on which $f$ is not integrable.

   Suppose $a < c < d < b$. We will show that $f$ is integrable on each of the subintervals, $[a, c]$, $[c, d]$, $[d, b]$ (which covers all possible types of closed subintervals). Since $f$ is integrable on $[a, b]$, given any $\varepsilon > 0$ we can find a partition $P = \{t_0, \ldots, t_n\}$ such that

   $$U(f, P) - L(f, P) < \varepsilon\,.$$

   Now let $Q$ be the partition of $[a, b]$ that contains all the points of $P$ and (if they are not already in $P$) the points $c$ and $d$. Since $P \subseteq Q$, it follows that

   $$U(f, Q) - L(f, Q) \le U(f, P) - L(f, P) < \varepsilon\,.$$

   Since $Q$ contains $c$ and $d$, we can break it up in the three parts, $Q = Q_1 \cup Q_2 \cup Q_3$, where (for some $j, k \in \mathbb{N}$)

   $$\begin{aligned} Q_1 &= \{a, t_1, \ldots, t_{j-1}, c\}\,, \\ Q_2 &= \{c, t_{j+1}, \ldots, t_{k-1}, d\}\,, \\ Q_3 &= \{d, t_{k+1}, \ldots, t_{n-1}, b\}\,. \end{aligned}$$

   Consequently,

   $$\begin{aligned} U(f, Q) &= U(f, Q_1) + U(f, Q_2) + U(f, Q_3)\,, \\ L(f, Q) &= L(f, Q_1) + L(f, Q_2) + L(f, Q_3)\,, \end{aligned}$$

   and hence

   $$U(f, Q) - L(f, Q) = \big[U(f, Q_1) - L(f, Q_1)\big] + \big[U(f, Q_2) - L(f, Q_2)\big] + \big[U(f, Q_3) - L(f, Q_3)\big].$$

   But each of the terms in square brackets is non-negative, and hence each of these terms must itself be less than $\varepsilon$. Thus, we have found partitions ($Q_1$, $Q_2$ and $Q_3$) of $[a, c]$, $[c, d]$ and $[d, b]$, respectively, that ensure the difference between the upper and lower sums of $f$ for $Q_i$ is less than $\varepsilon$, *i.e.*, $f$ is, in fact, integrable on each of the three subintervals. □

2. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$.

   (a) Let $P$ be a partition of $[0, 1]$. Which of the following statements about $L(f, P)$ is true?

   - ● $L(f, P) = 0$ for all $P$;
   - ○ $L(f, P) > 0$ for all $P$;
   - ○ $L(f, P) > 0$ for some $P$, but not all $P$;
   - ○ $L(f, P)$ can not be determined for any $P$.

   Regardless of how $[0, 1]$ is partitioned, every subinterval $[t_{i-1}, t_i]$ contains irrational numbers, hence $m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} = 0$ for all $i$. Consequently, $L(f, P) = 0$ for any partition $P$.

(b) For convenience, denote $\inf\{U\} \equiv \inf\{U(f,P) : P \text{ a partition of } [0,1]\}$. Which of the following statements about $\inf\{U\}$ is correct?

- ○ $\inf\{U\} = 0$;
- ○ $0 < \inf\{U\} < \frac{1}{2}$;
- ● $\inf\{U\} = \frac{1}{2}$;
- ○ $\frac{1}{2} < \inf\{U\} < 1$;
- ○ $\inf\{U\} = 1$.

Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0,1]$ (so $t_0 = 0$ and $t_n = 1$). For any $i$, if $t_i \in \mathbb{Q}$ then $M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\} = f(t_i) = t_i$. On the other hand, if $t_i \notin \mathbb{Q}$ then—since $\mathbb{Q}$ is dense in $\mathbb{R}$—for all $\varepsilon > 0$ there exists $\delta$ such that $0 < \delta < \varepsilon$ and $t_i - \delta \in \mathbb{Q}$, and hence $f(t_i - \delta) = t_i - \delta > t_i - \varepsilon$. Hence $M_i = \sup\{f(x) : x \in [t_{i-1}, t_i]\} = t_i$. Thus, for any partition $P$ of $[0,1]$ we have

$$U(f,P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1}) = \sum_{i=1}^{n} t_i(t_i - t_{i-1})\,. \qquad (\heartsuit)$$

The contribution of the subinterval $[t_{i-1}, t_i]$ to $U(f,P)$ is $t_i(t_i - t_{i-1})$, which is the area of the rectangle of width $t_i - t_{i-1}$ and height $t_i$. But for all $x \in [0,1]$, $f(x) \le x$, so—intuitively—the integral cannot be greater than the area of the trapezoid formed by the points

$$\{(t_{i-1}, 0),\ (t_{i-1}, t_{i-1}),\ (t_i, t_i),\ (t_i, 0)\}\,.$$

The area of this trapezoid is the sum of the areas of a square and a triangle, namely

$$(t_i - t_{i-1})t_{i-1} + \frac{1}{2}(t_i - t_{i-1})^2\,. \qquad (\spadesuit)$$

Motivated by this geometric observation, consider the following purely algebraic argument that depends only on the fact that $t_i > t_{i-1}$ (which is true for any partition), <u>not</u> on any picture:

$$\begin{aligned}
t_i(t_i - t_{i-1}) &= [t_i - t_{i-1} + t_{i-1}](t_i - t_{i-1}) \\
&= (t_i - t_{i-1})^2 + t_{i-1}(t_i - t_{i-1}) \\
&\ge \frac{1}{2}(t_i - t_{i-1})^2 + t_{i-1}(t_i - t_{i-1}) \qquad \text{(compare } \spadesuit\text{)} \\
&= \frac{1}{2}(t_i^2 - 2t_i t_{i-1} + t_{i-1}^2) + t_{i-1}t_i - t_{i-1}^2 \\
&= \frac{1}{2}(t_i^2 + t_{i-1}^2) - t_{i-1}^2 \\
&= \frac{1}{2}(t_i^2 - t_{i-1}^2)
\end{aligned}$$

Inserting this inequality in ($\heartsuit$), we have

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{n} t_i(t_i - t_{i-1}) \\
&\ge \sum_{i=1}^{n} \left[\frac{1}{2}(t_i^2 - t_{i-1}^2)\right] \\
&= \frac{1}{2}\left[\left(t_n^2 - t_{n-1}^2\right) + \left(t_{n-1}^2 - t_{n-2}^2\right) + \cdots + \left(t_1^2 - t_0^2\right)\right] \\
&= \frac{1}{2}\left(t_n^2 - t_0^2\right) \\
&= \frac{1}{2}\left(1^2 - 0^2\right) = \frac{1}{2}
\end{aligned}$$

Thus, $U(f,P) \geq \frac{1}{2}$ for any partition $P$ of $[0,1]$. It therefore follows that

$$\inf\{U\} \geq \frac{1}{2}\,. \qquad (\clubsuit)$$

We must now show that $\inf\{U\} \leq \frac{1}{2}$. To that end, consider the particular partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$, *i.e.*, $t_i = \frac{i}{n}$. From ($\heartsuit$), we have

$$\begin{aligned} U(f,P_n) &= \sum_{i=1}^{n} t_i(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\Big(\frac{i}{n} - \frac{i-1}{n}\Big) \\ &= \sum_{i=1}^{n} \frac{i}{n}\Big(\frac{1}{n}\Big) = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}\,, \end{aligned}$$

which implies

$$\inf\{U(f,P_n) : n \in \mathbb{N}\} = \frac{1}{2}\,,$$

and hence

$$\inf\{U(f,P) : P \text{ a partition of } [0,1]\} \leq \frac{1}{2}\,. \qquad (\diamondsuit)$$

Combining ($\diamondsuit$) with ($\clubsuit$), we have $\inf\{U\} = \frac{1}{2}$. □

(c) Is $f$ integrable on $[0,1]$?

- ○ Yes;
- ● No.

From part (a), $L(f,P) = 0$ for all partitions $P$, so $\sup\{L(f,P)\} = 0$. Therefore,

$$\sup\{L(f,P)\} = 0 \neq \frac{1}{2} = \inf\{U(f,P)\},$$

so $f$ is not integrable. □

3. A function is said to be ***piecewise continuous*** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.*, the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval.

(a) Suppose $f : [a,b] \to \mathbb{R}$ is piecewise continuous. Prove whichever statement is true:

- ● $f$ is integrable;
- ○ $f$ is not necessarily integrable.

*Proof.* Any piecewise continuous function is bounded, so the infimum $m$ and supremum $M$ of $f$ exist on any subinterval of $[a,b]$, and hence $L(f,P)$ and $U(f,P)$ are well-defined for any partition $P$ of $[a,b]$.

Suppose first that $f$ has only one point of discontinuity in $[a,b]$, and that it is at the left endpoint $a$. As usual, write a generic partition of $[a,b]$ as $P = \{t_0, t_1, \ldots, t_n\}$, and write

$$\begin{aligned} m_i &= \inf\{f(x) : t_{i-1} \leq x \leq t_i\}\,, \\ M_i &= \sup\{f(x) : t_{i-1} \leq x \leq t_i\}\,. \end{aligned}$$

We will choose $t_1$ so that the contribution of the first subinterval to the upper and lower sums is as small as we like. Also, for any given partition $P$, we'll define $P' = P \setminus \{t_0\}$.

We will first consider the special case in which $f(x)$ is strictly positive and, moreover, bounded below by a positive number, say $K$. Thus, $0 < K \le f(x)$ for all $x \in [a,b]$, and hence $0 < m_i < M_i$ for any partition $P$. Note that this implies, in particular, that $1/M_i$ is well-defined, and $0 \le \frac{M_i - m_i}{M_i} \le 1$ for any $i$.

Given $\varepsilon > 0$, let

$$t_1 = a + \frac{\varepsilon}{2M_1}\,,$$

where the motivation for the factor $\frac{1}{2M_1}$ will become clear below (wlog[1] we assume $\varepsilon$ is small enough that $t_1 < b$). In addition, choose $t_2, \dots, t_n$ so that

$$U(f,P') - L(f,P') < \frac{\varepsilon}{2}\,,$$

which is possible because $f$ is continuous and hence integrable on $[t_1, t_n] = [t_1, b]$. Then

$$\begin{aligned}
U(f,P) - L(f,P) &= M_1(t_1 - t_0) + U(f,P') - \big(m_1(t_1 - t_0) + L(f,P')\big) \\
&= (M_1 - m_1)(t_1 - t_0) + U(f,P') - L(f,P') \\
&= (M_1 - m_1)\frac{\varepsilon}{2M_1} + U(f,P') - L(f,P') \\
&= \underbrace{\frac{M_1 - m_1}{M_1}}_{\le 1} \cdot \frac{\varepsilon}{2} + \underbrace{U(f,P') - L(f,P')}_{< \frac{\varepsilon}{2}} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

Therefore, $f$ is integrable on $[a,b]$. A similar argument shows that $f$ is integrable if it has a single discontinuity at the right endpoint $b$. More generally, if $f$ has finitely many points of discontinuity then we can segment the interval $[a,b]$ into subintervals, each of which contains at most one discontinuity. $f$ is therefore integrable on each of these subintervals, so the integral segmentation theorem implies $f$ is integrable on all of $[a,b]$. Thus, any piecewise continuous function that is bounded below by a positive number is integrable.

Now consider the more general situation in which $f(x)$ need not be bounded below by a positive number. $f(x)$ is still bounded on $[a,b]$ (since it is piecewise continuous on $[a,b]$), so let $K = |\inf f(x) : x \in [a,b]| + 1$ and define $g(x) = f(x) + K$. Then $g(x)$ is piecewise continuous and satisfies $1 \le g(x)$, so it is integrable by our analysis above. But any constant function is also integrable, so by the algebra of integrals theorem, $f(x) = g(x) - K$ is integrable. □

(b) Recall that $\lceil x \rceil$ denotes the least integer that is greater than or equal to $x$. Let $f(x) = \lceil x \rceil$ for all $x \in \mathbb{R}$. Prove whichever of the following statements is true:

- ○ $\int_0^2 f = 0$;
- ○ $\int_0^2 f = 1$;
- ○ $\int_0^2 f = 2$;

[1] wlog = "without loss of generality".

- $\displaystyle\int_0^2 f = 3$;
- $\displaystyle\int_0^2 f$ does not exist, *i.e.*, $f$ is not integrable.

*Proof.* Since $\lceil x \rceil$ is piecewise continuous, part (a) implies that $f$ is integrable. We need to show specifically that $\int_0^2 f = 3$. Since $f$ is integrable on $[0,2]$, $\int_0^2 f$ exists and, for any partition $P$,

$$L(f,P) \ \le\ \int_0^2 f \ \le\ U(f,P)\,.$$

Given any $\varepsilon > 0$, we will construct a specific partition $P_\varepsilon$ such that

$$3-\varepsilon \ \le\ L(f,P_\varepsilon) \ \le\ \int_0^2 f \ \le\ U(f,P_\varepsilon) \ =\ 3\,,$$

which will imply the desired result because $\varepsilon > 0$ is arbitrary.
Thus, given $\varepsilon > 0$, define the partition $P_\varepsilon = \{t_0,t_1,t_2,t_3,t_4\}$ where

$$t_0 = 0, \quad t_1 = \frac{\varepsilon}{2}, \quad t_2 = 1, \quad t_3 = 1+\frac{\varepsilon}{2}, \quad t_4 = 2.$$

As usual, define $m_i$ and $M_i$ to be the infimum and supremum, respectively, of $f$ on $[t_{i-1},t_i]$, and note that if $x$ is an integer then $\lceil x \rceil = x$. We have

$$\begin{aligned} L(f,P) &= m_1(t_1-t_0)+m_2(t_2-t_1)+m_3(t_3-t_2)+m_4(t_4-t_3) \\ &= 0\cdot\tfrac{\varepsilon}{2}+1\big(1-\tfrac{\varepsilon}{2}\big)+1\cdot\tfrac{\varepsilon}{2}+2\big(1-\tfrac{\varepsilon}{2}\big) \\ &= 1-\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}+2-\varepsilon \\ &= 3-\varepsilon, \end{aligned}$$

and

$$\begin{aligned} U(f,P) &= M_1(t_1-t_0)+M_2(t_2-t_1)+M_3(t_3-t_2)+M_4(t_4-t_3) \\ &= 1\cdot\tfrac{\varepsilon}{2}+1\big(1-\tfrac{\varepsilon}{2}\big)+2\cdot\tfrac{\varepsilon}{2}+2\big(1-\tfrac{\varepsilon}{2}\big) \\ &= 1+2 \\ &= 3, \end{aligned}$$

as required. □

# Additional practice problems

4. Suppose $a < b$ and $f$ is integrable on $[a,b]$. Prove that

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx\,.$$

(The geometric interpretation should make this very plausible.) *Hint:* Every partition $P = \{t_0,\dots,t_n\}$ gives rise to a partition $P' = \{t_0+c,\dots,t_n+c\}$ of $[a+c,b+c]$, and conversely.

Let $g(x) = f(x-c)$ for all $x \in [a+c, b+c]$. Given a partition $P = \{t_0, \ldots, t_n\}$ of $[a,b]$, let $P' = \{t_0 + c, \ldots, t_n + c\}$. Then $P'$ is a partition of $[a+c, b+c]$ and we have

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \qquad \text{where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\
&= \sum_{i=1}^{n} m_i(t_i + c - c - t_{i-1}) \qquad \text{where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\
&= \sum_{i=1}^{n} m_i\big((t_i + c) - (t_{i-1} + c)\big) \qquad \text{where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\
&= \sum_{i=1}^{n} m_i\big((t_i + c) - (t_{i-1} + c)\big) \qquad \text{where } m_i = \inf\{f(x-c) : x \in [t_{i-1} + c, t_i + c]\} \\
&= \sum_{i=1}^{n} m_i\big((t_i + c) - (t_{i-1} + c)\big) \qquad \text{where } m_i = \inf\{g(x) : x \in [t_{i-1} + c, t_i + c]\} \\
&= L(g, P') \,.
\end{aligned}$$

Thus, every lower sum of $f$ for a partition $P$ on $[a,b]$ corresponds to a lower sum of $g$ for a partition $P'$ on $[a+c, b+c]$ and vice versa. Consquently,

$$\sup\{L(f,P) : P \text{ a partition of } [a,b]\} = \sup\{L(g,P') : P' \text{ a partition of } [a+c,b+c]\} \,.$$

Similarly,

$$\inf\{U(f,P) : P \text{ a partition of } [a,b]\} = \inf\{U(g,P') : P' \text{ a partition of } [a+c,b+c]\} \,.$$

If $f$ is integrable on $[a,b]$, then $\sup\{L(f,P)\} = \inf\{U(f,P)\}$, from which it follows immediately from above that $\sup\{L(g,P')\} = \inf\{U(g,P')\}$, *i.e.*, $g$ is integrable on $[a+c, b+c]$. Moreover,

$$\int_a^b f(x)\,dx = \sup\{L(f,P)\} = \sup\{L(g,P')\} = \int_{a+c}^{b+c} g(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx \,.$$

□

5. Suppose $b > 0$ and $f(x) = x$ for all $x \in \mathbb{R}$. Prove, using either the sup = inf or $\varepsilon$-$P$ definition of the integral, that $f$ is integrable on $[0,b]$ and

$$\int_0^b f = \frac{b^2}{2} \,.$$

*Note:* This exercise should help you appreciate the Fundamental Theorem of Calculus.

*Proof.* To apply the $\varepsilon$-$P$ definition, we need to show that for any given $\varepsilon > 0$ there is a partition $P$ of $[0,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

Let $P_n = \{t_0, \ldots, t_n\}$ be a partition of $[0,b]$ into $n$ subintervals of equal length. Thus, $t_i = ib/n$ for each $i = 0, 1, \ldots, n$. In addition, since $f(x) = x$ is an increasing function, we have

$$m_i = \inf\{f(x) : t_{i-1} \le x \le t_i\} = f(t_{i-1}) = t_{i-1}$$

(and, similarly, $M_i = t_i$). Therefore,

$$\begin{aligned}
L(f,P_n) &= \sum_{i=1}^{n} t_{i-1}(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{(i-1)b}{n} \cdot \frac{b}{n} \\
&= \frac{b^2}{n^2} \sum_{i=1}^{n} (i-1) = \frac{b^2}{n^2} \sum_{i=0}^{n-1} i = \frac{b^2}{n^2} \cdot \frac{(n-1)n}{2} = \frac{b^2}{2} \cdot \frac{(n-1)}{n} \,.
\end{aligned}$$

Similarly,

$$U(f, P_n) = \frac{b^2}{2} \cdot \frac{(n+1)}{n},$$

and hence

$$U(f, P_n) - L(f, P_n) = \frac{b^2}{n}.$$

Thus, given $\varepsilon > 0$, choose $n$ large enough that $b^2/n < \varepsilon$. Then the partition $P = P_n$ satisfies $U(f, P) - L(f, P) < \varepsilon$, proving that $f$ is integrable according to the $\varepsilon$-$P$ definition. Moreover, since

$$\frac{b^2}{2} \cdot \frac{(n-1)}{n} \;=\; L(f, P_n) \;\le\; \int_0^b f \;\le\; U(f, P_n) \;=\; \frac{b^2}{2} \cdot \frac{(n+1)}{n}$$

for <u>all</u> $n$, it follows that $\displaystyle\int_0^b f = \frac{b^2}{2}$. □

*Note:* If you're not yet convinced of the power of the Fundamental Theorem of Calculus, try computing $\displaystyle\int_0^b x^2\,dx$ directly from the definition of the integral.

6. Answer (and justify your answers to) the following questions, bearing in mind that lower and upper sums are defined by partitioning a closed interval $[a, b]$ into <u>closed</u> subintervals, so adjacent subintervals have a point in common. (*<u>Note</u>:* The definitions of lower and upper sums, and the Partition Theorem, are your friends for this problem.)

   (a) Which functions have the property that every lower sum equals every upper sum?

   Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ satisfies $L(f, P) = U(f, Q)$ for all partitions $P$ and $Q$. Then this true, in particular, for $P = Q = \{a, b\}$. But $L(f, \{a, b\}) = m(b - a)$ where $m = \inf\{f(x) : x \in [a, b]\}$ and $U(f, \{a, b\}) = M(b - a)$ where $M = \sup\{f(x) : x \in [a, b]\}$. Therefore $m = M$, *i.e.,* $f$ is constant. □

   (b) Which functions have the property that some upper sum equals some lower sum? (*<u>Note</u>:* The upper and lower sums could be calculated for different partitions.)

   Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Suppose $P_1$ and $P_2$ are particular partitions of $[a, b]$ with the property that $L(f, P_1) = U(f, P_2)$. Consider the partition $P = P_1 \cup P_2$. We know from the partition lemma and the partition theorem that

   $$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2).$$

   But $L(f, P_1) = U(f, P_2)$ by hypothesis, so it follows that $L(f, P) = U(f, P)$. Thus, we have established that if some upper sum equals some lower sum then, in fact, there exists a single partition $P$ of $[a, b]$ such that the upper and lower sums for $P$ are equal. Write $P = \{t_0, t_1, \ldots, t_n\}$ as usual, and let $m_i$ and $M_i$ be the greatest lower and least upper bounds for $f$ on the closed subintervals $[t_{i-1}, t_i]$, as usual. By definition, we always have $m_i \le M_i$ for $i = 1, \ldots, n$, so

   $$m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1}), \qquad i = 1, \ldots, n. \qquad (*)$$

   Suppose $m_j < M_j$ for some $j$. Then summing the $n$ inequalities (*) we get $L(f, P) < U(f, P)$, which is a contradiction. Therefore, we must have $m_i = M_i$ for all $i$, *i.e.,* $f$ is constant on each subinterval $[t_{i-1}, t_i]$. But adjacent subintervals have a point in common, so if $f$ constant on all subintervals, the constant value must be the same on each subinterval, *i.e.,* $f$ is a constant function on the entire interval $[a, b]$. □

(c) Which continuous functions have the property that all lower sums are equal?

Since all lower sums are equal, they are equal to $L(f, \{a, b\}) = m(b - a)$, where $m = \inf\{f(x) : x \in [a, b]\}$. Suppose that $f$ is not a constant function. Then, since $m$ is a lower bound for $f$ on $[a, b]$, there exists $u \in [a, b]$ such that $f(u) > m$. Consequently, since $f$ is continuous, the neighbourhood sign lemma (applied to the point $u$) implies that we can choose some partition $P = \{t_0, t_1, \ldots, t_n\}$ such that $f(x) > m$ on a subinterval $[t_{i-1}, t_i]$. But then $L(f, P) > m(b - a)$. $\Rightarrow\Leftarrow$ Hence $f$ must be constant. □

(d) (**Warning: much more challenging**) Which integrable functions have the property that all lower sums are equal?

*Hint:* A set $S$ is ***dense*** in $[a, b]$ if every open subinterval of $[a, b]$ contains a point of $S$. Begin by showing that if $f$ is integrable on $[a, b]$ and all lower sums are equal then $f(x) = m$ on a dense subset of $[a, b]$ (where $m = \inf\{f(x) : x \in [a, b]\}$).

Here is a sketch of one way to attack this problem.

(1) Let

$$m = \inf\{f(x) : x \in [a, b]\}.$$

Consider the coarsest partition $P_0 = \{a, b\}$. Show that

$$L(f, P_0) = m(b - a).$$

Conclude that if *all* lower sums are equal, then

$$L(f, P) = m(b - a) \quad \text{for every partition } P.$$

(2) Let $[c, d] \subset [a, b]$ and set $m_{[c,d]} = \inf\{f(x) : x \in [c, d]\}$. Show that $m_{[c,d]} = m$ for *every* subinterval $[c, d]$. (Hint: if $m_{[c,d]} > m$, compute $L(f, P)$ for the partition $P = \{a, c, d, b\}$ and compare with $m(b - a)$.)

(3) Now use integrability. Prove the following standard theorem:

If $f$ is Riemann integrable on $[a, b]$, then $f$ is continuous on a dense subset of $[a, b]$.

[That's actually a challenging problem on its own. It follows from the oscillation criterion for integrability (Theorem 8.16 in TBB) together with the oscillation characterization of continuity (Theorem 6.25 in TBB).]

(4) Let $x_0$ be any point where $f$ is continuous. Use Step (2) to show that for every $\varepsilon > 0$ and every $\delta > 0$ there exists $x \in (a, b)$ with $|x - x_0| < \delta$ and $f(x) < m + \varepsilon$. Combine this with continuity at $x_0$ to prove that $f(x_0) = m$.

(5) Conclude that $f(x) = m$ on a dense subset of $[a, b]$ (namely, on the dense set of continuity points of $f$). This proves the hint.

(6) Finally, prove the converse: if $f(x) = m$ on a dense subset of $[a, b]$, then for every partition $P$ one has $L(f, P) = m(b - a)$, so all lower sums are equal.

(7) Thus, the result is: If $f$ is integrable then all lower sums are equal if and only if $f$ equals its infimum $m$ on a dense subset of $[a, b]$.

*Note.* If you do not require integrability and just want to characterize when all lower sums of a *bounded* function are equal, the result is simply: if $m = \inf\{f(x) : x \in [a, b]\}$, then all lower sums are equal if and only if

$$\inf\{f(x) : x \in [c, d]\} = m \quad \text{for every subinterval } [c, d] \subseteq [a, b].$$