Mathematics 3A03 Real Analysis I

Winter 2025 ASSIGNMENT 2

Topic: The (Riemann/Darboux) Integral

Participation deadline: **Monday 3 February 2025 at 11:25am**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ is integrable on the closed interval $[a, b]$. Then:
   - ○ f is necessarily integrable on any closed subinterval of $[a, b]$;
   - ○ There might exist a closed subinterval of $[a, b]$ on which f is not integrable.

2. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$.
   (a) Let P be a partition of $[0, 1]$. Which of the following statements about $L(f, P)$ is true?
   - ○ $L(f, P) = 0$ for all P;
   - ○ $L(f, P) > 0$ for all P;
   - ○ $L(f, P) > 0$ for some P, but not all P;
   - ○ $L(f, P)$ can not be determined for any P.

   (b) For convenience, denote $\inf\{U\} \equiv \inf\{U(f, P) : P$ a partition of $[0, 1]\}$. Which of the following statements about $\inf\{U\}$ is correct?
   - ○ $\inf\{U\} = 0$;
   - ○ $0 < \inf\{U\} < \frac{1}{2}$;
   - ○ $\inf\{U\} = \frac{1}{2}$;
   - ○ $\frac{1}{2} < \inf\{U\} < 1$;
   - ○ $\inf\{U\} = 1$.

   (c) Is f integrable on $[0, 1]$?
   - ○ Yes;
   - ○ No.

3. A function is said to be ***piecewise continuous*** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.*, the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval.
   (a) Suppose $f : [a, b] \to \mathbb{R}$ is piecewise continuous. Prove whichever statement is true:
   - ○ f is integrable;
   - ○ f is not necessarily integrable.

   (b) Recall that $\lceil x \rceil$ denotes the least integer that is greater than or equal to x. Let $f(x) = \lceil x \rceil$ for all $x \in \mathbb{R}$. Prove whichever of the following statements is true:
   - ○ $\displaystyle\int_0^2 f = 0$;

- ○ $\int_0^2 f = 1$;
- ○ $\int_0^2 f = 2$;
- ○ $\int_0^2 f = 3$;
- ○ $\int_0^2 f$ does not exist, *i.e.*, f is not integrable.

Additional practice problems

4. Suppose $a < b$ and f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx\,.$$

(The geometric interpretation should make this very plausible.) *Hint:* Every partition $P = \{t_0, \ldots, t_n\}$ gives rise to a partition $P' = \{t_0 + c, \ldots, t_n + c\}$ of $[a+c, b+c]$, and conversely.

5. Suppose $b > 0$ and $f(x) = x$ for all $x \in \mathbb{R}$. Prove, using either the sup = inf or ε-P definition of the integral, that f is integrable on $[0, b]$ and

$$\int_0^b f = \frac{b^2}{2}\,.$$

Note: This exercise should help you appreciate the Fundamental Theorem of Calculus.

6. Answer (and justify your answers) to the following questions, bearing in mind that lower and upper sums are defined by partitioning a closed interval $[a, b]$ into closed subintervals, so adjacent subintervals have a point in common. (*Note:* The definitions of lower and upper sums, and the Partition Theorem, are your friends for this problem.)

   (a) Which functions have the property that every lower sum equals every upper sum?

   (b) Which functions have the property that some upper sum equals some lower sum? (*Note:* The upper and lower sums could be calculated for different partitions.)

   (c) Which continuous functions have the property that all lower sums are equal?

   (d) (**Warning: much more challenging**) Which integrable functions have the property that all lower sums are equal?

   Hint: A set S is ***dense*** in $[a, b]$ if every open subinterval of $[a, b]$ contains a point of S. Begin by showing that if f is integrable on $[a, b]$ and all lower sums are equal then $f(x) = m$ on a dense subset of $[a, b]$ (where $m = \inf\{f(x) : x \in [a, b]\}$).

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