Mathematics 3A03 Real Analysis I
Winter 2025 ASSIGNMENT 1
INSTRUCTOR'S SOLUTIONS
Topic: The Derivative
Participation deadline: **Monday 20 January 2025 @ 11:25am**

The meaning of the participation deadline is that you must answer the multiple choice questions on `childsmath` before that deadline in order to receive participation credit for the assignment. The `childsmath` poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on `childsmath`. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on `crowdmark`. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on `crowdmark` for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via `crowdmark`. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. A function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all x, and odd if $f(-x) = -f(x)$ for all x. Suppose f is differentiable. Which of the following statements are true?

☐ If f is even then f' is even.

■ If f is even then f' is odd.

☐ If f is odd then f' is odd.

■ If f is odd then f' is even.

Hint: You can prove or disprove the results directly from the definition of the derivative.

If f is differentiable and even then

$$\begin{aligned}
f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(-(x-h)) - f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} && f \text{ is even} \\
&= \lim_{h\to 0} \frac{f(x+(-h)) - f(x)}{h} \\
&= \lim_{h\to 0} -\frac{f(x+(-h)) - f(x)}{-h} \\
&= -\lim_{h\to 0} \frac{f(x+(-h)) - f(x)}{-h} \\
&= -\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && h \text{ can be either sign} \\
&= -f'(x)
\end{aligned}$$

so f' is odd. Similarly, if f is differentiable and odd then

$$\begin{aligned}
f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\
&= \lim_{h\to 0} \frac{f(-(x-h)) - f(-x)}{h} \\
&= \lim_{h\to 0} \frac{-f(x-h) + f(x)}{h} && f \text{ is odd} \\
&= \lim_{h\to 0} \frac{-f(x+(-h)) - (-f(x))}{h} \\
&= \lim_{h\to 0} -\frac{f(x+(-h)) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+(-h)) - f(x)}{-h} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && h \text{ can be either sign} \\
&= f'(x)
\end{aligned}$$

so f' is even. ☐

2. In class, we stated Rolle's Theorem as follows:

   If f is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b)$, then there exists $x \in (a,b)$ such that $f'(x) = 0$.

   Do we definitely need all three hypotheses of the theorem in order to be sure that the conclusion follows? Put another way, in which of the following cases is it possible to construct a function that satisfies the two conditions listed but for which it is not true that there exists $x \in (a,b)$ such that $f'(x) = 0$?

   - ■ f is continuous on $[a,b]$ and differentiable on (a,b);
     Let $a = 1$, $b = 2$, and $f(x) = x$. Then f is continuous on $[a,b]$ and diffentiable on (a,b), but $f(1) \neq f(2)$ and there is no $x \in (a,b)$ such that $f'(x) = 0$. □

   - ■ f is continuous on $[a,b]$ and $f(a) = f(b)$;
     Let $a = -1$, $b = 1$, and $f(x) = |x|$. Then f is continuous on $[a,b]$ and $f(a) = f(b)$, but f is not differentiable on (a,b), since it has no derivative at 0. There is no point $x \in (a,b)$ where $f'(x) = 0$. □

   - ■ f is differentiable on (a,b) and $f(a) = f(b)$.
     Let $a = 0$, $b = 1$, and $f(x) = \begin{cases} f(x) = \frac{1}{x} & x \neq 0 \\ 1 & x = 0. \end{cases}$. Then f is differentiable on (a,b) and $f(a) = f(b)$, but f is not continuous on $[a,b]$, since it is discontinuous at 0. There is no point $x \in (a,b)$ where $f'(x) = 0$. □

   In each case, either prove that the conclusion of Rolle's theorem still follows, or prove that it does not necessarily follow (*i.e.,* give an example of a function that satisfies the stated conditions but not the conclusion of Rolle's Theorem).

3. Which of the following statements are true?

   - ■ If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.
     Let a and b be any two points in the interval I, with $a < b$. By the Mean Value Theorem (MVT), $\exists\, \xi \in (a,b)$ such that
     $$\frac{f(b) - f(a)}{b - a} = f'(\xi)\,. \qquad (\clubsuit)$$
     But $f'(x) = 0$ for <u>all</u> $x \in I$, so in particular $f'(\xi) = 0$, and hence $f(b) = f(a)$. Since a and b were arbitrary points in the interval, f has the same value at every point in the interval. □.

   - ■ If f and g are defined on the same interval and $f'(x) = g'(x)$ for all x in the interval, then there is some $c \in \mathbb{R}$ such that $f = g + c$.
     Let $h = f - g$. Then h is differentiable and $h'(x) = 0$ for all x in the interval. Therefore, by part (a), h is constant, *i.e.,* $f = g + c$ for some $c \in \mathbb{R}$. □

   - ■ If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.
     Suppose $a, b \in I$ and $a < b$. By MVT $\exists\, \xi \in (a,b)$ such that ($\clubsuit$) holds. But $f'(x) > 0$ for all $x \in I$, so $f'(\xi) > 0$. Therefore $f(b) - f(a) > 0$, *i.e.,* $f(b) > f(a)$. Since this is true for any $a, b \in I$ with $a < b$, f is increasing on I. □

4. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and that $f'(x) \neq 0$ for all $x \in (a, b)$. It follows that:

- ○ f is increasing
- ○ f is decreasing
- ● f is monotone
- ○ f is not monotone
- ○ f has exactly one extremum

Proof. Pick $x_0 \in (a, b)$. Since $f'(x_0) \neq 0$, there are two cases:

Case 1: $f'(x_0) > 0$
Case 2: $f'(x_0) < 0$

We will show that Case 1 implies that f is strictly increasing; the proof that Case 2 implies that f is strictly decreasing is similar (or just replace f with $-f$ to reduce Case 2 to Case 1).

Assuming Case 1, we will first show that $f'(x) > 0$ for all $x \in (a, b)$. To see this, suppose not. Then there is some $x \in (a, b)$ with $f'(x) \leq 0$. Note that 0 is between $f'(x)$ and $f'(x_0)$. Darboux's theorem tells us that f' satisfies the Intermediate Value Property, so there is some c between x and x_0 with $f'(c) = 0$, which is a contradiction because $f'(x) \neq 0$ for all $x \in (a, b)$.

Now we prove that f is increasing. (Note this is now the same as the last part of the previous question, but we'll add a little more detail to the argument.) For this, fix $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is differentiable on (a, b) it is continuous on $[x_1, x_2] \subset (a, b)$ and differentiable on (x_1, x_2). By the MVT, there is some $c \in (x_1, x_2)$ so that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

We know that $f'(c) > 0$ (since we just showed that $f'(x) > 0$ for all $x \in (a, b)$), and we know that $x_2 - x_1 > 0$ (since we chose $x_1 < x_2$), so the above equation implies that

$$f(x_2) - f(x_1) = f'(x)(x_2 - x_1) > 0.$$

That is, $f(x_1) < f(x_2)$, which is what we wanted to show. □

Note that we proved this in class, but the way the argument is expressed above is different than the (shorter) proof in class.

Additional practice problems

5. (Cauchy Mean Value Theorem) Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Prove that there is some $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)\,.$$

Hint: Construct a function $h(x)$ to which you can apply Rolle's Theorem.

Proof. Let

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)\,.$$

Then h is continuous on $[a, b]$ and differentiable on (a, b) since both f and g are. In addition,

$$h(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) = [f(b)]g(a) - [g(b)]f(a)$$

and

$$\begin{aligned} h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) = [-f(a)]g(b) - [-g(a)]f(b) \\ &= f(b)[g(a)] - g(b)[f(a)], \end{aligned}$$

so $h(a) = h(b)$. Hence h satisfies the hypotheses of Rolle's theorem, which implies that there exists $x \in (a, b)$ such that $h'(x) = 0$. From the algebra of derivatives, it follows that

$$[f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x) = 0,$$

as required. □

6. (*Trapping principle.*) [***Warning: this is a hard problem.***] In class we considered the example of a function f defined in a neighbourhood I of 0 with the property that $|f(x)| \le x^2$ for all $x \in I$. We showed that any such f is differentiable at 0 and $f'(0) = 0$. Suppose, more generally, that there is some function g defined on I such that $|f(x)| \le g(x)$ for all $x \in I$.

   (a) Suppose $g(0) = 0$. What additional condition(s) on g are sufficient to guarantee that f is necessarily differentiable at 0? Propose and prove the most general theorem you can, *i.e.,* try to find the weakest sufficient additional condition(s) on g to ensure that $f'(0)$ exists.

   Since $|f(x)| \le g(x)$ for all $x \in I$, we must have $g(x) \ge 0$ for all $x \in I$. Suppose

   $$g \text{ is differentiable at } 0 \text{ and } g'(0) = 0, \qquad (\spadesuit)$$

   i.e.,

   $$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = 0.$$

   Since differentiability implies continuity, we also know g is continuous at 0; given $g(0) = 0$ this means

   $$\lim_{x \to 0} g(x) = g(0) = 0.$$

   In addition, $g(0) = 0$ implies $f(0) = 0$, so for all $x \neq 0$ we have

   $$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| \le \left|\frac{g(x)}{x}\right| = \left|\frac{g(x) - g(0)}{x - 0}\right|.$$

   From this, for all $x \neq 0$ we have

   $$-\left|\frac{g(x) - g(0)}{x - 0}\right| \le \frac{f(x) - f(0)}{x - 0} \le \left|\frac{g(x) - g(0)}{x - 0}\right|$$

   Here, both the LHS and RHS $\to 0$ as $x \to 0$ (because the quantity inside the absolute value bars $\to 0$ as $x \to 0$). Consequently, the squeeze theorem implies that the quantity in the middle $\to 0$ as $x \to 0$, *i.e.,*

   $$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0,$$

   i.e., f is differentiable at 0 and $f'(0) = 0$.

(b) Are the sufficient condition(s) you found in part (a) also necessary?

We must have g differentiable at 0. If not, then the special case of $f = g$ satisfies $|f(x)| \le g(x)$, yet f is not differentiable at 0. Given $g(0) = 0$, since $g(x) \ge 0$ for all x, 0 is a minimum point for g. Therefore, since g is differentiable at 0, we must have $g'(0) = 0$. Thus, given $g(0) = 0$, condition (♠) is both necessary and sufficient to ensure that f is differentiable at 0 and $f'(0) = 0$.

(c) What can be said if $g(0) \neq 0$? In particular, are the sufficient condition(s) you found still sufficient? If they were necessary with $g(0) = 0$, are they still necessary if $g(0) \neq 0$?

If $g(0) < 0$ then no function f can satisfy $|f(0)| \le g(0)$, so we must have $g(0) \ge 0$. If $g(0) > 0$, then (♠) is not sufficient. For example, consider $g(x) \equiv 1$. This g is differentiable everywhere and $g'(0) = 0$, yet any function f bounded by ± 1 satisfies $|f(x)| \le g(x)$.

The argument in part (b) showed that it is necessary that g is differentiable at 0, regardless of the value of $g(0)$. So suppose g is differentiable at 0 but $g(0) > 0$. In order to infer that any f satisfying $|f(x)| \le g(x)$ for all x is differentiable at 0, is it necessary that $g'(0) = 0$, as in condition (♠)?

Since g is differentiable at 0, it is continuous at 0. Therefore, since $g(0) > 0$, the neighbourhood sign lemma (stated and proved here in case it is not familiar from Math 3IA)

Neighbourhood Sign Lemma. Suppose I is an interval and $f : I \to \mathbb{R}$ is continuous at $a \in I$. If $f(a) > 0$ then f is positive in a neighbourhood of a. Similarly, if $f(a) < 0$, then f is negative in a neighbourhood of a.

Proof. Consider the case $f(a) > 0$. Since f is continuous at a, given $\varepsilon > 0$ $\exists \delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. Since $f(a) > 0$ we can take $\varepsilon = f(a)$. Thus, $\exists \delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < f(a)$, *i.e.*,

$$|x - a| < \delta \implies -f(a) < f(x) - f(a) < f(a) \implies 0 < f(x) < 2f(a).$$

In particular, $f(x) > 0$ in a neighbourhood[1] of radius δ about a.
The case $f(a) < 0$ is similar: take $\varepsilon = -f(a)$. □

implies that g is positive in some neighbourhood of 0. In fact, a slight modification of the proof of the neighbourhood sign lemma (take $\varepsilon = f(a)/2$ rather than $f(a)$) implies that $\exists m > 0$ and $\varepsilon > 0$ such that $g(x) > m$ for all $x \in (-\varepsilon, \varepsilon)$. Now consider any f satisfying $|f(x)| \le g(x)$ for all x and, moreover, $|f(x)| \le m$ for all $x \in (-\varepsilon, \varepsilon)$. An example of such of function is $f(x) = 0$ if $x \neq 0$ and $f(0) = m/2$, which is not differentiable at 0. So, in fact, no further condition on g can force all f's that satisfy $|f(x)| \le g(x)$ for all x to be differentiable at 0. In particular, the sign of $g'(0)$ is irrelevant (so certainly not necessary). □

Version of January 21, 2025 @ 17:33

[1] The neighbourhood is $(a - \delta, a + \delta)$, unless a is an endpoint of the set on which f is defined, in which case the neighbourhood is either $[a, a + \delta)$ or $(a - \delta, a]$.