Mathematics 3A03 Real Analysis I
Fall 2019 ASSIGNMENT 1 (Solutions)

This assignment was **due** on **Tuesday 17 September 2019 at 2:25pm** via crowdmark.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. Prove or disprove: $\sqrt{3/2}$ is irrational.

   Solution: Suppose, in order to derive a contradiction, that $\sqrt{3/2}$ is rational. Then $\exists m, n \in \mathbb{N}$ such that $\gcd(m,n) = 1$ and

$$\begin{aligned}
\sqrt{\frac{3}{2}} = \frac{m}{n} &\implies \frac{3}{2} = \frac{m^2}{n^2} \\
&\implies 3n^2 = 2m^2 \\
&\implies 3n^2 \text{ is even} \\
&\implies n^2 \text{ is even} \qquad (\because 3 \times (2k+1) = 6k+3 = 2(3k+1)+1, \text{ which is odd}) \\
&\implies n \text{ is even} \qquad (\because (2k+1)^2 = 4k^2+4k+1 = 2(k^2+2k)+1, \text{ which is odd}) \\
&\implies n = 2k \quad \text{for some } k \in \mathbb{N} \\
&\implies 3(2k)^2 = 2m^2 \\
&\implies 4 \times 3k^2 = 2m^2 \\
&\implies 2 \times 3k^2 = m^2 \\
&\implies m \text{ is even.}
\end{aligned}$$

   Thus, m and n are both even, contradicting $\gcd(m,n) = 1$. Therefore, $\sqrt{3/2}$ is not rational. □

2. What is wrong with the following "proof"? Let $x = y$. Then

$$\begin{aligned}
x^2 &= xy, \\
x^2 - y^2 &= xy - y^2, \\
(x+y)(x-y) &= y(x-y), \\
x+y &= y, \\
2y &= y, \\
2 &= 1.
\end{aligned}$$

   Solution: In the 4th line, we divided by zero (since $x = y$). □

3. Prove the following:

   (a) $|x-y| \le |x| + |y|$. (Give a very short proof.)
   Solution: From the triangle inequality, we have
   $|x-y| = |x+(-y)| \le |x| + |-y| = |x| + |y|$. □

(b) $|x| - |y| \leq |x - y|$. (A very short proof is possible, if you write things in the right way.)

Solution: $|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|$. $\square$

(c) $|(|x| - |y|)| \leq |x - y|$. (Why does this follow immediately from (3b)?)

Solution: If $|x| \geq |y|$ then (3b) implies $|(|x| - |y|)| \leq |x - y|$, whereas if $|x| \leq |y|$ then (3b) implies $|y| - |x| \leq |y - x|$, *i.e.*, $-(|x| - |y|) \leq |x - y|$, *i.e.*, $|(|x| - |y|)| \leq |x - y|$. $\square$

Further note: Since it will be useful in the next part, note that if x and y have opposite signs then the inequality is strict, *i.e.*, $|(|x| - |y|)| < |x - y|$. In particular, suppose $x > 0$ and $y < 0$. If $|x| \geq |y|$, then

$$||x| - |y|| = |x| - |y| < |x| + |y| = x + (-y) = |x + (-y)| = |x - y|,$$

whereas if $|x| < |y|$ then

$$||x| - |y|| = |y| - |x| < |y| + |x| = y + (-x) = |y + (-x)| = |y - x| = |x - y|.$$

(d) $|x + y + z| \leq |x| + |y| + |z|$. (Indicate when equality holds, and prove your statement.)

Solution: Applying the triangle inequality twice we have

$$|x + y + z| = |(x + y) + z| \leq |x + y| + |z| \leq |x| + |y| + |z|.$$

If $x, y, z \geq 0$ then $|x + y + z| = x + y + z = |x| + |y| + |z|$. Similarly, if $x, y, z \leq 0$ then $|x + y + z| = -(x + y + z) = (-x) + (-y) + (-z) = |x| + |y| + |z|$. Thus, equality holds if x, y, z are either all non-negative or all non-positive. Now suppose x, y, z are neither all non-negative nor all non-positive, which implies at least two of x, y, z are non-zero (why?), say $x, y \neq 0$. Suppose, in particular, that $x > 0$ and $y < 0$. Then

$$\begin{aligned} |x + y + z| &= |x - (-y) + z| \\ &\leq |x - (-y)| + |z| \qquad \text{(triangle inequality)} \\ &= ||x| - |y|| + |z| \\ &< |x - y| + |z| \qquad \text{(3c) with strict inequality} \because x, y \text{ have opposite signs} \\ &\leq |x| + |y| + |z| \qquad \text{(3a)}. \end{aligned}$$

Thus, equality holds if and only if x, y, z are all non-negative or all non-positive. $\square$

4. Prove by induction that if $x > -1$ then $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Solution: Let $P(n)$ be the proposition that "if $x > -1$ then $(1+x)^n \geq 1+nx$". $P(1)$ says that for $x > -1$, $(1+x)^1 \geq 1+1 \cdot x$. In this case, we actually have equality, so $P(1)$ is true. Now suppose $P(k)$ is true for some $k \in \mathbb{N}$, *i.e.*, if $x > -1$ then $(1+x)^k \geq 1+kx$. Then, given $x > -1$ (so $1 + x > 0$), we have

$$\begin{aligned} (1 + x)^{k+1} &= (1 + x)(1 + x)^k \\ &\geq (1 + x)(1 + kx) \qquad (\because x > -1 \text{ so we can apply } P(k)) \\ &= 1 + (k + 1)x + kx^2 \\ &\geq 1 + (k + 1)x \qquad (\because kx^2 \geq 0), \end{aligned}$$

so $P(k+1)$ is true. Hence, by the the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

5. For each of the following sets, find the greatest lower bound (inf), least upper bound (sup), minimum (min) and maximum (max), if they exist. If any of these do not exist, then indicate accordingly. Justify your assertions.

(a) $(1,2) \cup (2,3] \cup (-3,-2] \cup (-2,-1)$.

(b) $\{p^q : p, q \text{ prime}\}$.

(c) $\{x \in \mathbb{R} : x < 1/x\}$.

Solution: The answers to the questions are most easily summarized in a table:

	Set	inf	sup	min	max
(a)	$(1,2) \cup (2,3] \cup (-3,-2] \cup (-2,-1)$	-3	3	$\nexists$	3
(b)	$\{p^q : p, q \text{ prime}\}$	4	$\nexists$	4	$\nexists$
(c)	$\{x \in \mathbb{R} : x < 1/x\}$	$\nexists$	1	$\nexists$	$\nexists$

To justify the entries in this table, consider the following:

(a) The set is more transparently written as $(-3,-1) \cup (1,3] \setminus \{2\}$.

(b) The smallest prime is 2, so the minimum is $2^2 = 4$. There is no largest prime, so the set is not bounded above.

(c) Denote the set by E. We know $0 \notin E$ because the condition $x < 1/x$ is meaningless for $x = 0$. If $x > 0$ then $x \cdot x < x \cdot 1/x$, *i.e.*, $x^2 < 1$, so $x < 1$. If $x < 0$ then $x \cdot x > x \cdot 1/x$, *i.e.*, $x^2 > 1$, which implies (since $x < 0$) that $x < -1$. Thus $E = (-\infty, -1) \cup (0, 1)$. (If this is surprising to you, draw the functions $1/x$ and x.)

6. Suppose A and B are bounded subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded and $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

Solution: The intention of the problem was to consider *non-empty* sets A and B. So, we will assume $A \neq \varnothing \neq B$. Then, since A is bounded, it has a least upper bound ($\sup A$) and a greatest lower bound ($\inf A$), so we have

$$\inf A \leq a \leq \sup A \quad \forall a \in A.$$

Similarly, since B is bounded (and non-empty), we have

$$\inf B \leq b \leq \sup B \quad \forall b \in B.$$

Let $m = \min(\inf A, \inf B)$ and $M = \max(\sup A, \sup B)$ (note that m and M exist since we are just picking the minimum or maximum of two real numbers). It follows that

$$m \leq a \leq M \quad \forall a \in A,$$
$$m \leq b \leq M \quad \forall b \in B,$$

and hence

$$m \leq x \leq M \quad \forall x \in A \cup B.$$

Thus $A \cup B$ is bounded.

Note that $M = \sup(\sup A, \sup B)$, since the supremum and maximum are the same thing for a finite set of points. Moreover, either $\sup A \leq \sup B$ or $\sup B \leq \sup A$. Suppose that $\sup A \leq \sup B$, and hence that $M = \sup B$. Now suppose further, in order to derive a contradiction, that $\sup(A \cup B) < M$, *i.e.,* $\sup(A \cup B) < \sup B$. Then there must exist $b \in B$ such that $x < b$ for all $x \in A \cup B$. But $B \subseteq A \cup B$ so $b \in A \cup B$, and hence $b < b$. $\Rightarrow\Leftarrow$ The argument is similar if $\sup B \leq \sup A$, so we can conclude that $\sup(A \cup B) \geq M$. A similar argument allows us to rule out the possibility that $\sup(A \cup B) > M$, so we have finally that $\sup(A \cup B) = M = \sup\{\sup A, \sup B\}$, as required. $\square$