Mathematics 3A03 Real Analysis I

ASSIGNMENT 1

Topic: The Derivative

Participation deadline: **See course website**

The meaning of the participation deadline is that you must answer the multiple choice questions on childsmath before that deadline in order to receive participation credit for the assignment. The childsmath poll that you need to fill in for participation credit will be activated immediately after the last class before the above deadline.

Assignments in this course are graded only on the basis of participation, which you fulfill by answering the multiple choice questions on childsmath. You will get the same credit for any question that you answer, regardless of what your answer is. However, please answer the questions honestly so we obtain accurate statistics on how the class is doing.

You are encouraged to submit full written solutions on crowdmark. If you do so, you will not be graded on your work, but you will receive feedback that will hopefully help you to improve your mathematical skills and to prepare for the midterm test and the final exam.

There is no strict deadline for submitting written work on crowdmark for feedback, but please try to submit your solutions within a few days of the participation deadline so that the TA's work is spread out over the term. If you do not submit your solutions within a few days of the participation deadline then it may not be feasible for the TA to provide feedback via crowdmark. However, you can always ask for help with any problem during office hours with the TA or instructor.

You are encouraged to discuss and work on the problems jointly with your classmates, but remember that you will be working alone on the test and exam. You should attempt to solve the problems on your own before brainstorming with classmates, looking online, or asking the TA or instructor for help.

A full solution means either a proof or disproof of each statement that you are asked to consider when selecting your multiple choice answers.

Full solutions to the problems will be posted by the instructor. You should read the solutions only after doing your best to solve the problems, but do make sure to read the instructor's solutions carefully and ensure you understand them. If you notice any errors in the solutions, please report them to the instructor by e-mail.

Enjoy working on these problems!

– David Earn

1. A function $f : \mathbb{R} \to \mathbb{R}$ is even if $f(-x) = f(x)$ for all x, and odd if $f(-x) = -f(x)$ for all x. Suppose f is differentiable. Which of the following statements are true?

   ☐ If f is even then f' is even.

   ☐ If f is even then f' is odd.

   ☐ If f is odd then f' is odd.

   ☐ If f is odd then f' is even.

   Hint: You can prove or disprove the results directly from the definition of the derivative.

2. In class, we stated Rolle's Theorem as follows:

   > If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there exists $x \in (a, b)$ such that $f'(x) = 0$.

   Do we definitely need all three hypotheses of the theorem in order to be sure that the conclusion follows? Put another way, in which of the following cases is it possible to construct a function that satisfies the two conditions listed but for which it is not true that there exists $x \in (a, b)$ such that $f'(x) = 0$?

   ☐ f is continuous on $[a, b]$ and differentiable on (a, b);

   ☐ f is continuous on $[a, b]$ and $f(a) = f(b)$;

   ☐ f is differentiable on (a, b) and $f(a) = f(b)$.

   In each case, either prove that the conclusion of Rolle's theorem still follows, or prove that it does not necessarily follow (*i.e.,* give an example of a function that satisfies the stated conditions but not the conclusion of Rolle's Theorem).

3. Which of the following statements are true?

   ☐ If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

   ☐ If f and g are defined on the same interval and $f'(x) = g'(x)$ for all x in the interval, then there is some $c \in \mathbb{R}$ such that $f = g + c$.

   ☐ If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.

4. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and that $f'(x) \neq 0$ for all $x \in (a, b)$. It follows that:

   ○ f is increasing

   ○ f is decreasing

   ○ f is monotone

   ○ f is not monotone

   ○ f has exactly one extremum

Additional practice problems

5. (Cauchy Mean Value Theorem) Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Prove that there is some $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x) \,.$$

Hint: Construct a function $h(x)$ to which you can apply Rolle's Theorem.

6. (*Trapping principle.*) [***Warning: this is a hard problem.***] In class we considered the example of a function f defined in a neighbourhood I of 0 with the property that $|f(x)| \le x^2$ for all $x \in I$. We showed that any such f is differentiable at 0 and $f'(0) = 0$. Suppose, more generally, that there is some function g defined on I such that $|f(x)| \le g(x)$ for all $x \in I$.

   (a) Suppose $g(0) = 0$. What additional condition(s) on g are sufficient to guarantee that f is necessarily differentiable at 0? Propose and prove the most general theorem you can, *i.e.,* try to find the weakest sufficient additional condition(s) on g to ensure that $f'(0)$ exists.

   (b) Are the sufficient condition(s) you found in part (a) also necessary?

   (c) What can be said if $g(0) \neq 0$? In particular, are the sufficient condition(s) you found still sufficient? If they were necessary with $g(0) = 0$, are they still necessary if $g(0) \neq 0$?

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