Mathematics 3A03 Real Analysis I

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2019 ASSIGNMENT 6 (Solutions)

This assignment wass **due** on **Monday 8 April 2019 at 11:25am**.

1. Suppose f is continuous on $[a, b]$. Prove that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx\,.$$

Solution: Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$, let

$$m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\}\,,$$
$$m_i' = \inf\{|f(x)| : x \in [t_{i-1}, t_i]\}\,,$$

and define M_i and M_i' similarly using sup rather than inf. Then $m_i \le m_i'$ and $M_i \le M_i'$ for all i and hence

$$L(f, P) \le L(|f|\,, P)\,,$$
$$U(f, P) \le U(|f|\,, P)\,.$$

Since this is true for all partitions P, we have

$$\sup\{L(f, P)\} \le \sup\{L(|f|\,, P)\}$$
$$\inf\{U(f, P)\} \le \inf\{U(|f|\,, P)\}\,.$$

Now, since f is continuous, $|f|$ is also continuous. Moreover, since any continuous function is integrable, we have $\sup\{L(f, P)\} = \inf\{U(f, P)\} = \int_a^b f$ and $\sup\{L(|f|\,, P)\} = \inf\{U(|f|\,, P)\} = \int_a^b |f|$. Hence $\int_a^b f \le \int_a^b |f|$. □

2. Prove that if $f(x) = \displaystyle\int_0^x f(t)\,dt$ then $f = 0$.

Hint: First prove that f is differentiable and $f'(x) = f(x)$. Then consider the derivative of the function $g(x) = f(x)/e^x$.

Solution: First note that f is continuous everywhere since it is defined as the integral of a function (itself, as it happens). But then from the (First) Fundamental Theorem of Calculus, f is differentiable everywhere since is it the integral of a continuous function (itself, as it happens), and $f'(x) = f(x)$ for all x.

Now consider $g(x) = f(x)/e^x$, which is well defined because $e^x > 0$ for all $x \in \mathbb{R}$. Since f and exp are both differentiable for all x, we can use the quotient rule to obtain

$$g'(x) = \frac{e^x f'(x) - f(x)e^x}{e^{2x}} = \frac{e^x f(x) - f(x)e^x}{e^{2x}} = 0\,, \qquad \text{for all } x.$$

Therefore, (from 2016 Assignment 5) there exists $c \in \mathbb{R}$ such that $g(x) = c$, *i.e.*, $f(x) = ce^x$. But $f(0) = \int_0^0 f = 0$, hence $f = 0$. □

3. Consider the sequence of functions $\{f_n\}$, where

$$f_n(x) = \frac{1}{n(1+nx^2)}\,, \qquad x \in \mathbb{R}.$$

(a) For which $x \in \mathbb{R}$ does the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converge pointwise?

Solution: For $x = 0$ we have $\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. For any $x \neq 0$, we have

$$0 < \sum_{n=1}^{\infty} \frac{1}{n(1+nx^2)} < \sum_{n=1}^{\infty} \frac{1}{n(nx^2)} = \frac{1}{x^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\,,$$

which converges, so $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise (by the comparison test) for all $x \neq 0$. □

(b) For which $a, b \in \mathbb{R}$ $(a < b)$ does the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n$ converge uniformly on $[a, b]$ to a continuous function?

Solution: From part (a) we know that any interval $[a, b]$ on which the series of functions converges uniformly cannot include $x = 0$. Hence either $a > 0$ or $b < 0$. Suppose $a > 0$. Then let

$$M_n = \frac{1}{a^2 n^2}\,,$$

and note that $\sum_{n=1}^{\infty} M_n$ converges (*e.g.*, by the ratio test). Hence, by the Weierstrass M-test, $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a, b]$ to a function f, and since each f_n is continuous the uniform limit is also continuous. A similar argument establishes the result for $b < 0$, so the series converges uniformly to a continuous function on any closed interval not containing 0. □

(c) For which $a, b \in \mathbb{R}$ $(a < b)$ does the series of functions $\displaystyle\sum_{n=1}^{\infty} f_n$ converge uniformly on $[a, b]$ to a differentiable function f? For such a, b, is f' necessarily the uniform limit of $\displaystyle\sum_{n=1}^{\infty} f_n'$?

Solution: Again, since the series does not converge pointwise at $x = 0$, we must restrict attention to $a > 0$ or $b < 0$. Again, suppose $a > 0$. To apply the theorem on differentiability and uniform convergence, note that

- $f_n'(x) = -\dfrac{2x}{(1+nx^2)^2}$, so each f_n' is continuous on $[a, b]$ (in fact on $\mathbb{R}$).
- For $x \geq a > 0$,

$$|f_n'(x)| = \frac{2\,|x|}{(1+nx^2)^2} < \frac{2\,|x|}{(nx^2)^2} = \frac{2\,|x|}{n^2x^4} = \frac{2}{|x|^3}\cdot\frac{1}{n^2} \leq \frac{2}{a^3}\cdot\frac{1}{n^2}\,.$$

Hence, taking $M_n = 2/(a^3n^2)$, the Weierstrass M-test implies that $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$.

- We already know from part (b) that $\sum_{n=1}^{\infty} f_n$ converges pointwise to a continuous function f.

Consequently, if $a > 0$ then f is differentiable on $[a, b]$ and $f' = \sum_{n=1}^{\infty} f_n'$ (*i.e.*, f' is the uniform limit of $\sum_{n=1}^{\infty} f_n'$). In addition, the argument is similar if $b < 0$, so the result is true for any closed interval not containing 0. □

(d) Rather than closed, finite intervals $[a, b]$, consider infinite open intervals (a, ∞). Answer parts (b) and (c) again after revising them to read "For which $a \in \mathbb{R}$ does the series... converge uniformly on (a, ∞) to...".

Solution: The Weierstrass M-test relates to any domain $D \subseteq \mathbb{R}$. It is not necessary that D be a closed interval, or indeed any sort of interval. The arguments in parts (b) and (c) apply equally well to (a, ∞) for $a > 0$ or $(-\infty, b)$ for $b < 0$. Thus, the answer to the question posed is "any $a > 0$". □