Mathematics 3A03 Real Analysis I

http://www.math.mcmaster.ca/earn/3A03

2019 ASSIGNMENT 5 (Solutions)

This assignment was **due** on **Monday 25 March 2019 at 11:25am**.

1. Classify the discontinuities of the rational function

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-1}, & x \neq \pm 1, \\ c_1, & x = 1, \\ c_2, & x = -1. \end{cases}$$

Note: See the textbook (TBB, §5.9.1, p. 331) for the definitions of removable, jump and essential discontinuities.

Solution: For $x \neq \pm 1$, we have

$$f(x) = \frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1}\,, \qquad x \neq \pm 1.$$

The function $1/(x-1)$ is continuous except at $x = 1$ so f is certainly continuous for $x \neq \pm 1$. Moreover,

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{1}{x-1} = -\frac{1}{2}\,,$$

so if $c_2 = -\frac{1}{2}$ then f is also continuous at $x = -1$. If $c_2 \neq -\frac{1}{2}$ then f is discontinuous at $x = -1$, but since $f(x)$ does have a limit as $x \to -1$ this discontinuity is removable (simply change the value of c_2 to be $-\frac{1}{2}$). On the other hand,

$$\lim_{x \to 1^{\pm}} f(x) = \lim_{x \to 1^{\pm}} \frac{1}{x-1} = \pm\infty\,.$$

Since neither the right hand nor left hand limit exists, f has an essential discontinuity at $x = 1$. □

2. Suppose that f is a function on a closed domain D, and let $E = f(D)$ be the range of f. Prove that f is continuous on D *if and only if* the inverse image of every closed set is closed.

Note: The inverse image of a set A is the set of all points in the domain of f that are mapped into A, *i.e.*, $f^{-1}(A) = \{x \in D : f(x) \in A\}$.

Note: Problem 1(b) on 2016 Assignment 5 showed that a continuous function does not necessarily map closed sets to closed sets.

Solution: The key to the solution given here is to use the sequence definition of continuity.

$\Longrightarrow$ Suppose f is continuous on D, $A \subseteq E$ and A is closed. We must show that $f^{-1}(A)$ is a closed subset of D. If $f^{-1}(A)$ has no accumulation points then it is closed and we are done. Therefore, assume the set of accumulation points of $f^{-1}(A)$ is non-empty and let x_* be an accumulation point of $f^{-1}(A)$. Since D is closed we know $x_* \in D$. Moreover, since x_* is an accumulation point of $f^{-1}(A)$, there is a sequence $\{x_n\} \subseteq f^{-1}(A) \subseteq D$ that converges to x_*. But f is continuous on D, so $x_n \to x_*$ implies $f(x_n) \to f(x_*)$. Now, since $\{f(x_n)\}$ is a sequence in $f(f^{-1}(A)) = A$, and A is closed, we must have $f(x_*) \in A$. Hence $x_* \in f^{-1}(A)$. Therefore, since x_* was an arbitrary accumulation point of $f^{-1}(A)$, we have established that $f^{-1}(A)$ is closed.

$\Longleftarrow$ Suppose $f : D \to E$ and the inverse image under f of every closed subset of E is a closed subset of D. We must show that f is continuous on D. Let $\{x_n\} \subseteq D$ be a sequence that converges to a point x_*. Since D is closed, we must have $x_* \in D$. Now consider the sequence $\{f(x_n)\} \subseteq E$. We must show that this sequence converges to $f(x_*)$. Suppose not. Then there exists $\varepsilon > 0$, and a subsequence $\{f(x_{n_k})\}$ such that $|f(x_{n_k}) - f(x_*)| > \varepsilon$ for all $k \in \mathbb{N}$. We now need to use the fact that the inverse image under f of every closed set in E is closed in D. We do not know if the set $\{f(x_{n_k})\} \subseteq E$ is closed; however, its closure $\overline{\{f(x_{n_k})\}}$ is certainly closed, and yet $\overline{\{f(x_{n_k})\}}$ must not contain $f(x_*)$ because each point $f(x_{n_k})$ of $\{f(x_{n_k})\}$ is more than a distance ε from $f(x_*)$. But the inverse image $f^{-1}(\overline{\{f(x_{n_k})\}})$ is closed by assumption, and it is always true that $\{f(x_{n_k})\} \subseteq \overline{\{f(x_{n_k})\}}$, so we certainly have

$$f^{-1}(\overline{\{f(x_{n_k})\}}) \supseteq f^{-1}(\{f(x_{n_k})\}) \supseteq \{x_{n_k}\}\,.$$

Now note that $x_{n_k} \to x_*$ implies $x_* \in \overline{\{x_{n_k}\}}$, and since the set $f^{-1}(\overline{\{f(x_{n_k})\}})$ on the left above is closed and contains $\{x_{n_k}\}$, it must also contain the accumulation point x_*. Thus, $x_* \in f^{-1}(\overline{\{f(x_{n_k})\}})$ and hence $f(x_*) \in f(f^{-1}(\overline{\{f(x_{n_k})\}})) = \overline{\{f(x_{n_k})\}}$, contradicting our inference above that $f(x_*) \notin \overline{\{f(x_{n_k})\}}$. $\Rightarrow\Leftarrow$ Thus, f must indeed be continuous. □

Note: This problem is much easier if you start from the fact, proved in 2017 Assignment 5 Problem 1, that a function is continuous if and only if the inverse image of every open set is open.

3. Suppose f and g are continuous on $[a,b]$ and differentiable on (a,b). Prove that there is some $x \in (a,b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)\,.$$

Hint: Construct a function $h(x)$ to which you can apply Rolle's Theorem.

Solution: Let

$$h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)\,.$$

Then h is continuous on $[a,b]$ and differentiable on (a,b) since both f and g are. In addition,

$$h(a) = [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) = [f(b)]g(a) - [g(b)]f(a)$$

and

$$h(b) = [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) = [-f(a)]g(b) - [-g(a)]f(b)$$
$$= f(b)[g(a)] - g(b)[f(a)]\,,$$

so $h(a) = h(b)$. Hence h satisfies the hypotheses of Rolle's theorem, which implies that there exists $x \in (a, b)$ such that $h'(x) = 0$. From the algebra of derivatives, it follows that

$$[f(b) - f(a)]g'(x) - [g(b) - g(a)]f'(x) = 0\,,$$

as required. □

4. Answer (and justify your answers) to the following questions, bearing in mind that lower and upper sums are defined by partitioning a closed interval $[a, b]$ into closed subintervals, so adjacent subintervals have a point in common. (*Note:* The definitions of lower and upper sums, and the Partition Theorem, are your friends for this problem.)

   (a) Which functions have the property that every lower sum equals every upper sum?

   Solution: Suppose $a < b$ and $f : [a, b] \to \mathbb{R}$ satisfies $L(f, P) = U(f, Q)$ for all partitions P and Q. Then this true, in particular, for $P = Q = \{a, b\}$. But $L(f, \{a, b\}) = m(b-a)$ where $m = \inf\{f(x) : x \in [a, b]\}$ and $U(f, \{a, b\}) = M(b-a)$ where $M = \sup\{f(x) : x \in [a, b]\}$. Therefore $m = M$, *i.e.,* f is constant. □

   (b) Which functions have the property that some upper sum equals some lower sum? (*Note:* The upper and lower sums could be calculated for different partitions.)

   Solution: Let $a < b$ and $f : [a, b] \to \mathbb{R}$. Suppose P_1 and P_2 are particular partitions of $[a, b]$ with the property that $L(f, P_1) = U(f, P_2)$. Consider the partition $P = P_1 \cup P_2$. We know from the partition lemma and the partition theorem that

$$L(f, P_1) \le L(f, P) \le U(f, P) \le U(f, P_2)\,.$$

   But $L(f, P_1) = U(f, P_2)$ by hypothesis, so it follows that $L(f, P) = U(f, P)$. Thus, we have established that if some upper sum equals some lower sum then, in fact, there exists a single partition P of $[a, b]$ such that the upper and lower sums for P are equal. Write $P = \{t_0, t_1, \ldots, t_n\}$ as usual, and let m_i and M_i be the greatest lower and least upper bounds for f on the closed subintervals $[t_{i-1}, t_i]$, as usual. By definition, we always have $m_i \le M_i$ for $i = 1, \ldots, n$, so

$$m_i(t_i - t_{i-1}) \le M_i(t_i - t_{i-1})\,, \qquad i = 1, \ldots, n. \qquad (*)$$

   Suppose $m_j < M_j$ for some j. Then summing the n inequalities (*) we get $L(f, P) < U(f, P)$, which is a contradiction. Therefore, we must have $m_i = M_i$ for all i, *i.e.,* f is constant on each subinterval $[t_{i-1}, t_i]$. But adjacent subintervals have a point in common, so if f constant on all subintervals, the constant value must be the same on each subinterval, *i.e.,* f is a constant function on the entire interval $[a, b]$. □

(c) Which continuous functions have the property that all lower sums are equal?

Solution: Since all lower sums are equal, they are equal to $L(f, \{a, b\}) = m(b-a)$, where $m = \inf\{f(x) : x \in [a, b]\}$. Suppose that f is not a constant function. Then, since m is a lower bound for f on $[a, b]$, there exists $u \in [a, b]$ such that $f(u) > m$. Consequently, since f is continuous, the neighbourhood sign lemma (applied to the point u) implies that we can choose some partition $P = \{t_0, t_1, \dots, t_n\}$ such that $f(x) > m$ on a subinterval $[t_{i-1}, t_i]$. But then $L(f, P) > m(b - a)$. $\Rightarrow\Leftarrow$ Hence f must be constant. $\square$

(d) (**Bonus**) Which integrable functions have the property that all lower sums are equal?

Hint: First show that if f is integrable on $[a, b]$ and all lower sums are equal then $f(x) = m$ on a dense subset of $[a, b]$ (where $m = \inf\{f(x) : x \in [a, b]\}$).

5. Suppose $a < b$ and f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x - c)\,dx\,.$$

(The geometric interpretation should make this very plausible.) *Hint:* Every partition $P = \{t_0, \dots, t_n\}$ gives rise to a partition $P' = \{t_0 + c, \dots, t_n + c\}$ of $[a + c, b + c]$, and conversely.

Solution: Let $g(x) = f(x-c)$ for all $x \in [a+c, b+c]$. Given a partition $P = \{t_0, \dots, t_n\}$ of $[a, b]$, let $P' = \{t_0 + c, \dots, t_n + c\}$. Then P' is a partition of $[a + c, b + c]$ and we have

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^{n} m_i(t_i - t_{i-1}) \qquad \text{where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\
&= \sum_{i=1}^{n} m_i(t_i - c + c - t_{i-1}) \qquad \text{where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\
&= \sum_{i=1}^{n} m_i\big((t_i - c) - (t_{i-1} - c)\big) \qquad \text{where } m_i = \inf\{f(x) : x \in [t_{i-1}, t_i]\} \\
&= \sum_{i=1}^{n} m_i\big((t_i - c) - (t_{i-1} - c)\big) \qquad \text{where } m_i = \inf\{f(x - c) : x \in [t_{i-1} + c, t_i + c]\} \\
&= \sum_{i=1}^{n} m_i\big((t_i - c) - (t_{i-1} - c)\big) \qquad \text{where } m_i = \inf\{g(x) : x \in [t_{i-1} + c, t_i + c]\} \\
&= L(g, P')\,.
\end{aligned}$$

Thus, every lower sum of f for a partition P on $[a, b]$ corresponds to a lower sum of g for a partition P' on $[a + c, b + c]$ and vice versa. Consquently,

$$\sup\{L(f, P) : P \text{ a parition of } [a, b]\} = \sup\{L(g, P') : P' \text{ a parition of } [a + c, b + c]\}\,.$$

Similarly,

$$\inf\{U(f, P) : P \text{ a parition of } [a, b]\} = \inf\{U(g, P') : P' \text{ a parition of } [a + c, b + c]\}\,.$$

If f is integrable on $[a, b]$, then $\sup\{L(f, P)\} = \inf\{U(f, P)\}$, from which it follows immediately from above that $\sup\{L(g, P')\} = \inf\{U(g, P')\}$, *i.e.*, g is integrable on $[a + c, b + c]$. Moreover,

$$\int_a^b f(x)\, dx = \sup\{L(f, P)\} = \sup\{L(g, P')\} = \int_{a+c}^{b+c} g(x)\, dx = \int_{a+c}^{b+c} f(x - c)\, dx\,.$$

$\square$