Mathematics 3A03 Real Analysis I

http://www.math.mcmaster.ca/earn/3A03

2019 ASSIGNMENT 5

This assignment is **due** on **Monday 25 March 2019 at 11:25am**.
PLEASE NOTE that you must **submit online** via crowdmark.
You will receive an e-mail from crowdmark with the required link.
Do **NOT** submit a hardcopy of this assignment.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. Classify the discontinuities of the rational function

$$f(x) = \begin{cases} \dfrac{x+1}{x^2-1}, & x \neq \pm 1, \\ c_1, & x = 1, \\ c_2, & x = -1. \end{cases}$$

   Note: See the textbook (TBB, §5.9.1, p. 331) for the definitions of removable, jump and essential discontinuities.

2. Suppose that f is a function on a closed domain D, and let $E = f(D)$ be the range of f. Prove that f is continuous on D *if and only if* the inverse image of every closed set is closed.

   Note: The inverse image of a set A is the set of all points in the domain of f that are mapped into A, *i.e.*, $f^{-1}(A) = \{x \in D : f(x) \in A\}$.

   Note: Problem 1(b) on 2016 Assignment 5 showed that a continuous function does not necessarily map closed sets to closed sets.

3. Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Prove that there is some $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)\,.$$

   Hint: Construct a function $h(x)$ to which you can apply Rolle's Theorem.

4. Answer (and justify your answers) to the following questions, bearing in mind that lower and upper sums are defined by partitioning a closed interval $[a, b]$ into closed subintervals, so adjacent subintervals have a point in common. (*Note:* The definitions of lower and upper sums, and the Partition Theorem, are your friends for this problem.)

   (a) Which functions have the property that every lower sum equals every upper sum?

   (b) Which functions have the property that some upper sum equals some lower sum? (*Note:* The upper and lower sums could be calculated for different partitions.)

(c) Which continuous functions have the property that all lower sums are equal?

(d) (**Bonus**) Which integrable functions have the property that all lower sums are equal?

5. Suppose $a < b$ and f is integrable on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx\,.$$

(The geometric interpretation should make this very plausible.) *Hint:* Every partition $P = \{t_0, \ldots, t_n\}$ gives rise to a partition $P' = \{t_0 + c, \ldots, t_n + c\}$ of $[a + c, b + c]$, and conversely.