Mathematics 3A03 Real Analysis I

http://www.math.mcmaster.ca/earn/3A03

2019 ASSIGNMENT 4 (Solutions)

This assignment is **due** on **Friday 8 March 2019 at 1:25pm**.
PLEASE NOTE that you must **submit online** via crowdmark.
You will receive an e-mail from crowdmark with the required link.
Do **NOT** submit a hardcopy of this assignment.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. Give an example of a sequence of closed sets F_1, F_2, F_3, $\ldots$, whose union is neither open nor closed. Can this be achieved with a sequence that contains only finitely many distinct sets?

   Solution: Let $F_n = [\frac{1}{n}, 1]$. Note that each F_n is closed and consider the sequence of sets $\{F_n : n \in \mathbb{N}\}$. Then
   $$\bigcup_{n=1}^{\infty} F_n = (0,1]\,,$$
   which is neither open nor closed.

   It is not possible to achieve this with a sequence containing only finitely many distinct sets, since any finite union of closed sets is closed. □

2. Suppose that $E \subseteq \mathbb{R}$, $K \subseteq \mathbb{R}$, E is closed and K is compact. Show that $E \cap K$ is compact, by proving directly that $E \cap K$ satisfies each of the following equivalent properties:

   (a) closed and bounded;

   Solution: Recall that a set is closed iff it contains all its accumulation points. Suppose x is an accumulation point of $E \cap K$. Then, by definition, every deleted neighbourhood of x contains a point of $E \cap K$. But since every point of $E \cap K$ is in both E and K, it follows that every deleted neighbourhood of x contains a point of E and a point of K. Hence x is an accumulation point of E and of K, and since both E and K are closed, we must have $x \in E$ and $x \in K$, *i.e.*, $x \in E \cap K$. Thus $E \cap K$ is closed.

   Now, since K is bounded, there exists $M > 0$ such that $|x| < M$ for all $x \in K$. But if $y \in E \cap K$ then $y \in K$, so $|y| < M$, *i.e.*, $E \cap K$ is bounded.

   Thus, $E \cap K$ is both closed and bounded. □

   (b) Bolzano-Weierstrass property;

   Solution: We must show that any sequence in $E \cap K$ contains a subsequence that converges to a point in $E \cap K$. Let $\{x_n\}$ be a sequence in $E \cap K$. Since $E \cap K \subseteq K$, $\{x_n\}$ is a sequence in K. But K is compact so $\{x_n\}$ contains a subsequence $\{x_{n_m}\}$ that converges to a point $x \in K$, *i.e.*, $\exists x \in K$ such that, $\forall \varepsilon > 0$, $\exists M \in \mathbb{N} \ni \forall m \geq M$, $|x - x_{n_m}| < \varepsilon$. In particular, taking $\varepsilon = \varepsilon_\ell = \frac{1}{\ell}$, for each $\ell \in \mathbb{N}$ we can find $x_\ell \in \{x_{n_m}\} \subseteq E \cap K \subseteq E$ such that $|x - x_\ell| < \frac{1}{\ell}$. Hence x is an accumulation point of E. But E is closed, so $x \in E$. Thus, the arbitrary sequence $\{x_n\} \subseteq E \cap K$ contains a subsequence that converges to a point $x \in E \cap K$. $\square$

(c) Heine-Borel property.

Solution: We must show that any open cover of $E \cap K$ contains a finite subcover of $E \cap K$. Let $\mathcal{U}$ be an open cover of $E \cap K$. $\mathcal{U}$ is not necessarily an open cover of K unless $K \subseteq E$. However, since E is closed, it follows that E^c is open, and E^c certainly covers all points of K that are not also points of E. Therefore, $\mathcal{V} = \mathcal{U} \cup E^c$ is an open cover of K. But K is compact, so $\mathcal{V}$ contains a finite subcover of K, say $\mathcal{W} = \{U_1, \ldots, U_n\}$. It follows that $\mathcal{W}$ also covers $E \cap K \subseteq K$. Moreover, if $E^c \in \mathcal{W}$, it can be discarded for the purpose of covering $E \cap K$, since $(E \cap K) \cap E^c = \varnothing$. We therefore have a finite subcollection of the original open cover $\mathcal{U}$ that covers $E \cap K$. $\square$

3. For which of the following functions f is there a continuous function g with domain $\mathbb{R}$ such that $g(x) = f(x)$ for all x in the domain of f?

(i) $f(x) = \dfrac{x^2 - 4}{x - 2}$,

Solution: $\mathrm{dom} f = \mathbb{R} \setminus \{2\}$. Noting that for all $x \in \mathrm{dom} f$, we have

$$\frac{x^2 - 4}{x - 2} = \frac{(x-2)(x+2)}{x - 2} = x + 2\,,$$

we see that $g(x) = x + 2$ agrees with f at every $x \in \mathrm{dom} f$, but g is continuous at all $x \in \mathbb{R}$. $\square$

(ii) $f(x) = \dfrac{|x|}{x}$,

Solution: $\mathrm{dom} f = \mathbb{R} \setminus \{0\}$. Note that for all $x \neq 0$ we have

$$f(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0, \end{cases}$$

from which it follows that

$$\lim_{x \to 0^-} f(x) = -1 \neq 1 = \lim_{x \to 0^+} f(x)\,.$$

Consequently, no function g that agrees with f on its domain can be continuous at 0. $\square$

(iii) $f(x) = 0$, x irrational.

Solution: Let $g(x) = 0$ for all $x \in \mathbb{R}$. Then $g(x) = f(x)$ for all $x \in \mathrm{dom} f = \mathbb{R} \setminus \mathbb{Q}$, and g is continous on $\mathbb{R}$. $\square$.

4. Prove that if f is continuous at a, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $|x - a| < \delta$ and $|y - a| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

Solution: Since f is continuous at a, given $\varepsilon > 0$ we can find $\delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \frac{\varepsilon}{2}$. If both $|x - a| < \delta$ and $|y - a| < \delta$ then $|f(x) - f(a)| < \frac{\varepsilon}{2}$ and $|f(y) - f(a)| < \frac{\varepsilon}{2}$, and hence

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(a) + f(a) - f(y)| \\ &= \big|f(x) - f(a) - \big(f(y) - f(a)\big)\big| \\ &\le |f(x) - f(a)| + |f(y) - f(a)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,, \end{aligned}$$

as required. □

5. Suppose $a, b \in \mathbb{R}$ and $a < b$. Prove directly from the definition that $f(x) = x^2$ is uniformly continuous on the closed interval $[a, b]$. Is f uniformly continuous on the open interval (a, b)?

Solution: We must show that given any $\varepsilon > 0$, we can find $\delta > 0$ such that if $x, y \in [a, b]$ and $|x - y| < \delta$ then $|x^2 - y^2| < \varepsilon$.

Note first that $|x^2 - y^2| = |x - y|\,|x + y|$. We can make $|x - y|$ as small as we like by choosing δ sufficiently small. The challenge is to bound $|x + y|$. But note that for any $x, y \in [a, b]$ we have

$$|x + y| \le |x| + |y| \le 2\max\big(\,|a|\,, |b|\,\big)\,.$$

Therefore, let $M = 2\max(|a|\,, |b|)$ and note that $M > 0$ (since $a < b$, so it is impossible that both a and b vanish).

Now, given $\varepsilon > 0$, let $\delta = \varepsilon/M$. Then, if $|x - y| < \delta$, and $a \le x, y \le b$, we have

$$\left|x^2 - y^2\right| = |x - y|\,|x + y| < \frac{\varepsilon}{M} \cdot M = \varepsilon\,,$$

as required.

Finally, a function that is uniformly continuous on a given domain is certainly uniformly continuous on any subset of the domain. Consequently, given that x^2 is uniformly continuous on $[a, b]$, it follows immediately that it is uniformly continuous on (a, b). However, a uniformly continuous function on an open interval (a, b) need not even be continuous on $[a, b]$. □