Mathematics 3A03 Real Analysis I

http://www.math.mcmaster.ca/earn/3A03

2019 ASSIGNMENT 3 (Solutions)

This assignment was **due** on **Friday 15 Feb 2019 at 1:25pm** via crowdmark.

Note: Not all questions will be marked. The questions to be marked will be determined after the assignment is due.

1. Let $\{x_n\}$ be a bounded sequence and let $x = \sup\{x_n : n \in \mathbb{N}\}$. Suppose that, moreover, $x_n < x$ for all n. Prove that there is a subsequence of $\{x_n\}$ that converges to x.

   Solution: We need to show that we can pick a subsequence of $\{x_{n_j}\} \subseteq \{x_n\}$, such that $x_{n_j} \to x$ as $j \to \infty$, *i.e.*, we must demonstrate the existence of a subsequence $\{x_{n_j}\}$ with the property that $\forall \varepsilon > 0\ \exists J \in \mathbb{N}$ such that $\forall j \geq J$, $x - x_{n_j} < \varepsilon$.

   Since x is the least upper bound of $\{x_n\}$, given any $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $0 < x - x_n < \varepsilon$. Intuitively, we can construct the required subsequence by finding, for each $j \in \mathbb{N}$, an element of the original sequence that is within a distance $1/j$ from the least upper bound x. However, we need to make sure that we don't pick the same x_n multiple times (for example we can't have $n_1 = n_2$, because we would then be choosing the same sequence element twice). We can ensure we get a subsequence via the following inductive construction.

   Let $n_1 = 1$, so the first element of $\{x_{n_j}\}$ is x_1 (the first element of the original sequence $\{x_n\}$). Then, given $n_1, n_2, \ldots, n_{j-1}$, choose n_j such that $x - x_{n_j} < x - x_{n_{j-1}}$ (to ensure we're picking a new sequence element) and $x - x_{n_j} < \frac{1}{j}$ (to ensure $x_{n_j} \to x$). Put another way, we choose n_j, such that

   $$x - x_{n_j} < \varepsilon_j \overset{\textbf{def}}{=} \min\left\{x - x_{n_{j-1}},\quad \frac{1}{j}\right\}.$$

   Then, given any $\varepsilon > 0$, if we choose $J > 1/\varepsilon$ then we can be sure that $x - x_{n_j} < \varepsilon$ for any $j \geq J$ as required. $\square$

2. Show directly that the sequence $s_n = \frac{1}{n}$ is a Cauchy sequence.

   Solution: We need to show that given any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$,

   $$\left|\frac{1}{m} - \frac{1}{n}\right| < \varepsilon.$$

   Note from the triangle inequality that $\forall m, n \in \mathbb{N}$,

   $$\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} \leq 2\max\left\{\frac{1}{m},\ \frac{1}{n}\right\} = \frac{2}{\min(m,n)}. \qquad (*)$$

   Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $2/N < \varepsilon$, *i.e.*, choose $N \in \mathbb{N}$ such that $N > 2/\varepsilon$. Then for any $m, n \geq N$, $\min(m,n) \geq N$, so $2/\min(m,n) \leq 2/N < \varepsilon$, and hence from (*) we have $|(1/m) - (1/n)| < \varepsilon$ as required. $\square$

3. Show directly that if $\{s_n\}$ is a Cauchy sequence then so too is $\{|s_n|\}$. From this conclude that $\{|s_n|\}$ converges whenever $\{s_n\}$ converges.

Solution: Suppose $\{s_n\}$ is Cauchy, *i.e.,* $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall m, n \geq N$, $|s_n - s_m| < \varepsilon$. Note first from the triangle inequality that for any $x, y \in \mathbb{R}$, we have

$$|x| - |y| \leq |x - y| \ ,$$

and similarly,

$$|y| - |x| \leq |y - x| = |x - y| \ ,$$

and hence

$$||x| - |y|| \leq |x - y| \ .$$

Therefore, given $\varepsilon > 0$ find $N \in \mathbb{N}$ such that $\forall m, n \geq N$, $|s_n - s_m| < \varepsilon$ (which is possible because $\{s_n\}$ is Cauchy). Then

$$||s_n| - |s_m|| \leq |s_n - s_m| < \varepsilon \,,$$

i.e., $\{|s_n|\}$ is Cauchy. □

4. Define a sequence $\{a_n\}$ recursively by setting $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that $\{a_n\}$ converges and find its limit. *Hint:* First show by induction that a_n is bounded.

Solution: Let's begin by showing, specifically, that $1 \leq a_n < 2$ for all $n \in \mathbb{N}$. We know $a_1 = 1$ and $1 \leq 1 < 2$, so the base case of our induction is true. Now suppose $1 \leq a_n < 2$. We know $a_{n+1} = \sqrt{1 + a_n}$, so $a_{n+1} > 1$ and

$$a_{n+1}^2 = 1 + a_n \leq 1 + 2 < 4 \,,$$

from which we have $1 < a_{n+1} < 2$. Hence, by the principle of mathematical induction, $\{a_n\}$ is bounded (specifically, $1 \leq a_n < 2$ for all $n \in \mathbb{N}$).

If we can now show that $\{a_n\}$ is monotonic, then the monotone convergence theorem will guarantee that it converges. To that end, let's try to show that $a_{n+1} \geq a_n$ for all n. Observe that

$$\begin{aligned} a_{n+1} \geq a_n &\iff \sqrt{1 + a_n} \geq a_n \\ &\iff 1 + a_n \geq a_n^2 \qquad (\because a_n \geq 1) \\ &\iff a_n^2 - a_n - 1 \leq 0 \\ &\iff \left(a_n - \frac{1 + \sqrt{5}}{2}\right)\left(a_n - \frac{1 - \sqrt{5}}{2}\right) \leq 0 \\ &\Longleftarrow a_n \leq \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad a_n \geq \frac{1 - \sqrt{5}}{2} \\ &\Longleftarrow 1 \leq a_n < 2 \,. \end{aligned}$$

Hence our bounds actually imply the sequence is monotonic, and hence it converges by the monotone convergence theorem.

Finally, since the sequence converges, if we denote its limit by a then we must have $a = \sqrt{1+a}$, *i.e.,* $a^2 - a - 1 = 0$. Since $a > 1$, we have

$$a = \frac{1+\sqrt{5}}{2}\,.$$

This is the golden ratio, which turns up in many places. $\square$

5. Suppose $A \subset B \subset \mathbb{R}$. Prove that if B is countable then A is countable.

   Solution: If $A = \varnothing$ then A is countable, so assume $A \neq \varnothing$. We will prove that A is the range of a sequence. Since B is countable, it is the range of a sequence $\{f(n)\}$, *i.e.,* $f : \mathbb{N} \to B$ is surjective (onto B). Define a sequence $g(n)$ recursively as follows. Since $\varnothing \neq A \subset B$, the well-ordering of the natural numbers implies that there is a *least* natural number i such that $f(i) \in A$. Let $g(1) = g(2) = \cdots = g(i) = f(i)$ and for all $n > i$ let

$$g(n) = \begin{cases} f(n) & \text{if } f(n) \in A\,, \\ g(n-1) & \text{if } f(n) \notin A\,. \end{cases}$$

   Then $\{g(n)\} \subset A$. Moreoever, since $A \subset B$ and f is *onto* B, every element of A occurs in the sequence $\{f(n)\}$. Hence $g(\mathbb{N}) = \{g(n)\} = A$. $\square$

6. Suppose $A \subset \mathbb{R}$ and $B = \{b : b = a \text{ or } b = a^2 \text{ for some } a \in A\}$. Prove that if A is countable then B is countable.

   Solution: If $A = \varnothing$ then $B = \varnothing$, which is countable, so assume $A \neq \varnothing$. Since A is countable, there is a sequence whose range is A, *i.e.,* $\exists f : \mathbb{N} \to A$ such that $f(\mathbb{N}) = A$. Define a function $g : \mathbb{N} \to B$ via

$$g(n) = \begin{cases} f((n+1)/2) & \text{if } n \text{ is odd,} \\ [f(n/2)]^2 & \text{if } n \text{ is even.} \end{cases}$$

   Then the range of the sequence $\{g(n)\}$ is B, so B is countable. $\square$

7. Show (a) that no interior point of a set can be a boundary point, (b) that it is possible for an accumulation point to be a boundary point, and (c) that every isolated point must be a boundary point.

   Solution: Consider a set $E \subseteq \mathbb{R}$.

   (a) For any $x \in E^\circ$, there exists $c > 0$ such that $(x-c, x+c) \subseteq E$. But $(x-c, x+c)$ is therefore a neighbourhood of x that contains no points of $\mathbb{R} \setminus E$, *i.e.,* x is not a boundary point of E.

   (b) Suppose $E = (0, 1]$. The point $0 \notin E$, but 0 is an accumulation point of E since any neighbourhood of 0 contains points of E.

   (c) Suppose x is an isolated point of a set $E \subset \mathbb{R}$. Then there is a neighbourhood $(x-c, x+c)$ of x for which x is the only element of E. Any other neighbourhood $(x-d, x+d)$ of x contains x, and regardless of whether d is less than or greater than c, there are points of $(x-c, x+c) \setminus \{x\}$ in $(x-d, x+d)$, so x is a boundary point of E. $\square$

8. Express the closed interval $[0,1]$ as an intersection of a sequence of open sets. Can it also be expressed as a union of a sequence of open sets?

Solution: For the first part, note that

$$[0,1] = \bigcap_{n=1}^{\infty} \Big(-\frac{1}{n}, 1 + \frac{1}{n} \Big) .$$

For the second part, note that any union of open sets is open, yet $[0,1]$ is not open so it cannot be expressed as union of open sets. (Note that it is important to state that $[0,1]$ is "not open". The fact that $[0,1]$ is closed does not imply on its own that $[0,1]$ is not open; recall that $\mathbb{R}$ is both open and closed.)

To prove that the union of a sequence of open sets is open, let

$$\mathcal{U} = \bigcup_{n=1}^{\infty} U_n$$

be a union of a sequence of open sets $\{U_n\}$. If $x \in \mathcal{U}$ then there must be an open set U_i such that $x \in U_i$. But $U_i \subseteq \mathcal{U}$, which implies x is an interior point of $\mathcal{U}$, *i.e.,* $\mathcal{U}$ is open.

To prove more generally (and slightly more abstractly) that *any* union of open sets is open, suppose $\mathcal{U}$ is a union of open sets. If $x \in \mathcal{U}$ then there must be an open set $U \subseteq \mathcal{U}$ such that $x \in U$, which implies x is an interior point of $\mathcal{U}$, *i.e.,* $\mathcal{U}$ is open. □