Mathematics 3A03 Real Analysis I

http://www.math.mcmaster.ca/earn/3A03

2019 ASSIGNMENT 2 (Solutions)

This assignment was **due in the appropriate locker** on **Friday 1 Feb 2019 at 1:25pm**.

1. Did you know that all horses are the same colour?

   What is wrong with the following proof by induction?

   Theorem 1. *All horses are the same colour.*

   Proof. Let $P(n)$ be the proposition "Any n horses are the same colour."

   Consider the base case of one horse. It is obviously the same colour, so $P(1)$ is true.

   Now assume $P(k)$ is true and consider a collection of $k+1$ horses, which we will agree to label $h_1, h_2, \ldots, h_k, h_{k+1}$. Horses $h_1, \ldots, h_k$ form a collection of k horses, so by the induction hypothesis, they are all the same colour. Similarly, $h_2, \ldots, h_{k+1}$ form a collection of k horses, so they are all the same colour. But horses $h_2, \ldots, h_k$ are in both collections, so all $k+1$ horses must be the same colour!

   Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$, *i.e.,* all horses are the same colour. □

   With thanks to my undergraduate analysis tutor, Costa Roussakis, who first alerted me to this illuminating argument. –DE

   Solution: The error occurs almost immediately, when considering $P(1)$. What exactly does the author mean by one thing being "the same colour". The expression "the same colour" has meaning only when comparing *at least two* things. The base case for the proposition $P(n)$ is actually $n = 2$, not $n = 1$. Good luck proving $P(2)$!

   Another comment worth making—also made strongly to me by Costa Roussakis—is that you should always avoid "obviously", "as we all know", "it is self-evident that", or any similar phrases in your proofs. Such words or phrases make the reader feel like they must be stupid if they don't see why the claim is true, which discourages them from thinking carefully about it. Instead of saying something is obvious, explain why it is true so your argument can be dissected! You will certainly not get credit for saying something is "obvious" on your assignments, tests or exam. More importantly, in any context, it is always better to articulate arguments clearly and concisely than to say the conclusion is obvious.

2. Use the formal definition of a limit of a sequence to prove that

(a) $\lim_{n\to\infty} \frac{n+1}{n+2} = 1$;

Solution: Given $\varepsilon > 0$ we must show there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left|\frac{n+1}{n+2} - 1\right| < \varepsilon\,. \tag{$*$}$$

To see how to achieve this, consider that

$$\frac{n+1}{n+2} - 1 = \frac{n+2-1}{n+2} - 1 = 1 - \frac{1}{n+2} - 1 = -\frac{1}{n+2}\,,$$

so (*) is equivalent to

$$\frac{1}{n+2} < \varepsilon\,,$$

which is in turn equivalent to

$$n + 2 > \frac{1}{\varepsilon}\,.$$

Therefore, given $\varepsilon > 0$, choose (by the Archimedean property) $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Then $N + 2 > 1/\varepsilon$. Moreover, for any $n \geq N$ we also have $n + 2 > \varepsilon$, and hence $1/(n + 2) < \varepsilon$. From our calculation above, this proves that $\forall n \geq N$, $|(n + 2)/(n + 1) - 1| < \varepsilon$, as required. □

(b) $\lim_{n\to\infty} \frac{n^3-1}{n^4-1} = 0$;

Solution: Given $\varepsilon > 0$ we must show there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left|\frac{n^3-1}{n^4-1}\right| < \varepsilon\,.$$

First note that $\forall n > 1$

$$\frac{n^3-1}{n^4-1} = \frac{(n-1)(n^2+n+1)}{(n-1)(n^3+n^2+n+1)} = \frac{(n^2+n+1)}{(n^3+n^2+n+1)} \tag{1a}$$

$$< \frac{(n^2+n+1)}{n^3} = \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}\,. \tag{1b}$$

Also note that $\forall n > 1$

$$\frac{1}{n^3} < \frac{1}{n^2} < \frac{1}{n}\,.$$

Therefore, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $1/N < \varepsilon/3$, *i.e.*, $N > 3/\varepsilon$. Then, $\forall n \geq N$, we have $n > 3/\varepsilon$, *i.e.*, $1/n < \varepsilon/3$, and hence

$$\frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3} < \frac{1}{n} + \frac{1}{n} + \frac{1}{n} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

It then follows from (1) that $\forall n > 1$

$$0 < \frac{n^3-1}{n^4-1} < \varepsilon\,,$$

and hence

$$\left|\frac{n^3-1}{n^4-1}\right| < \varepsilon\,,$$

as required. □

(c) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$;

Solution: We must show that given any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N$

$$\frac{n!}{n^n} < \varepsilon\,.$$

To see this, note that for any $n > 5$ we can write

$$\begin{aligned}
\frac{n!}{n^n} &= \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n-2}{n} \cdot \frac{n-1}{n} \cdot \frac{n}{n} \\
&= \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n-2}{n} \cdot \frac{n-1}{n} \\
&< \frac{1}{n} \cdot \frac{2}{n} \cdots \frac{n-2}{n} \\
&\vdots \\
&< \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \\
&< \frac{1}{n} \cdot \frac{2}{n} \\
&< \frac{1}{n}\,.
\end{aligned}$$

Therefore, given $\varepsilon > 0$, choose a natural number $N > 1/\varepsilon$, *i.e.*, such that $1/N < \varepsilon$. Then $1/n < \varepsilon$ for any $n \geq N$, and hence by the sequence of inequalities above, it follows that $n!/n^n < \varepsilon \ \forall n \geq N$, as required. □

3. Use the formal definition to prove that the following sequences a_n diverge as $n \to \infty$.

(a) $a_n = 1 + nd$, $d \neq 0$.

Solution: Suppose $d > 0$. Given any $M \in \mathbb{R}$, find a natural number $N > M/d$ (by the Archimedean property). Then for any $n \geq N$ we have $a_n = 1 + nd \geq 1 + Nd > Nd > (M/d)d = M$. Hence a_n diverges to ∞. Similarly, for $d < 0$, $a_n \to -\infty$. □

(b) $a_1 = 1$, $a_{n+1} = 2^{a_n}$ for $n \in \mathbb{N}$.

Solution: Consider the first few sequence elements:

$$a_1 = 2^0 = 1, \quad a_2 = 2^1 = 2, \quad a_3 = 2^2, \quad a_4 = 2^{2^2}, \quad a_5 = 2^{2^{2^2}}, \quad \ldots$$

This appears to grow very quickly and the claim that $\{a_n\}$ diverges is certainly plausible. To prove formally that $a_n \to \infty$, we need to demonstrate that a_n can be made as large as we like. To that end, let's first prove that $a_n \geq n$ for all $n \in \mathbb{N}$, which can be established by induction. The base case is the statement $a_1 \geq 1$, which is true because we are told that, in fact, $a_1 = 1$. Now suppose $a_n \geq n$. Then $a_n - n \geq 0$ so $2^{a_n} = 2^{a_n - n + n} = 2^{a_n - n}2^n \geq 2^n$. Consequently,

$$a_{n+1} = 2^{a_n} \geq 2^n \geq n + 1\,,$$

as required. (The statement $2^n \geq n+1$ really needs proof as well, but this is easy to show by induction via $2^{n+1} = 2 \cdot 2^n \geq 2 \cdot n \geq n+1$.) To complete our proof, given $M \in \mathbb{R}$, choose a natural number $N > M$. Then $N+1 > M$ and hence $n+1 > M \; \forall n \geq N$. But our analysis above then implies $a_n \geq n+1 > M \; \forall n \geq N$. Since M was an arbitrary real number, we have $a_n \to \infty$. □

4. (a) Suppose that $\{a_n\}$ is a convergent sequence for which $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$ and $L = \lim_{n\to\infty} a_n$. Prove that $L \in [0,1]$.

   Solution: Since $0 \leq a_n \leq 1$, we know $\{a_n\}$ is bounded above by 1. Let $a = \sup\{a_n\}$. We must have $a \leq 1$ since 1 is an upper bound. Now suppose $L > 1$ and let $\varepsilon = (L-1)/2$. Since $a_n \to L$, $\exists N \in \mathbb{N}$ such that $|a_N - L| < \varepsilon$, *i.e.*, $-\varepsilon < a_N - L < \varepsilon$,

$$\begin{aligned} i.e., \qquad -\frac{L-1}{2} \; &< \; a_N - L \; < \; \frac{L-1}{2} \\ i.e., \qquad \frac{L+1}{2} \; &< \; a_N \; < \; \frac{3L-1}{2} \,. \end{aligned}$$

   But $L > 1$, so $(L+1)/2 > 1$, and hence $a_N > 1 \geq a$. Apparently a_N is larger than the least upper bound of $\{a_n\}$. $\Rightarrow\Leftarrow$ Therefore, the assumption that $L > 1$ must be false, *i.e.*, we must have $L \leq 1$. A similar argument shows $L \geq 0$. □

   (b) Find a convergent sequence $\{a_n\}$ of points all in $(0,1)$ such that $\lim_{n\to\infty} a_n$ is not in $(0,1)$.

   Solution: Let $a_n = \frac{1}{n+1}$. Then $0 < a_n < 1$ for all $n \in \mathbb{N}$, but $a_n \to 0 \notin (0,1)$. □

5. Show that the formal definition of convergence of a sequence that we gave in class is equivalent to the following slight modification:

   We write $\lim_{n\to\infty} s_n = L$ provided that for every positive integer m there is an integer N so that $|s_n - L| < \frac{1}{m}$ whenever $n \geq N$.

   Solution: The usual formal definition is

$$\lim_{n\to\infty} s_n = L \quad \overset{\textbf{def}}{=} \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \ni \quad n \geq N \implies |s_n - L| < \varepsilon \,.$$

   First suppose that $s_n \to L$ according to the "slight modification". To verify the usual definition, given $\varepsilon > 0$, choose (via the Archimedean property) $m \in \mathbb{N}$ such that $m > 1/\varepsilon$, *i.e.*, $1/m < \varepsilon$. Since $s_n \to L$ according to the "slight modification", we can choose $N \in \mathbb{N}$ such that $|s_n - L| < \frac{1}{m}$ for all $n \geq N$. Then, since $1/m < \varepsilon$, we have $|s_n - L| < \varepsilon$ for all $n \geq N$, as required.

   Now suppose that $s_n \to L$ according to the usual definition. To verify the "slight modification", given $m \in \mathbb{N}$, let $\varepsilon = 1/m$. Since $s_n \to L$ according to the the usual definition, choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $|s_n - L| < \varepsilon$. But since $\varepsilon = 1/m$, this immediately implies $|s_n - L| < \frac{1}{m} \; \forall n \geq N$, as required. □