Mathematics 3A03 Real Analysis I

http://www.math.mcmaster.ca/earn/3A03

2019 ASSIGNMENT 2

This assignment is **due in the appropriate locker** on **Friday 1 Feb 2019 at 1:25pm**.

1. Did you know that all horses are the same colour?

   What is wrong with the following proof by induction?

   Theorem 1. *All horses are the same colour.*

   Proof. Let $P(n)$ be the proposition "Any n horses are the same colour."

   Consider the base case of one horse. It is obviously the same colour, so $P(1)$ is true.

   Now assume $P(k)$ is true and consider a collection of $k+1$ horses, which we will agree to label $h_1, h_2, \ldots, h_k, h_{k+1}$. Horses $h_1, \ldots, h_k$ form a collection of k horses, so by the induction hypothesis, they are all the same colour. Similarly, $h_2, \ldots, h_{k+1}$ form a collection of k horses, so they are all the same colour. But horses $h_2, \ldots, h_k$ are in both collections, so all $k+1$ horses must be the same colour!

   Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$, *i.e.*, all horses are the same colour. □

   With thanks to my undergraduate analysis tutor, Costa Roussakis, who first alerted me to this illuminating argument. –DE

2. Use the formal definition of a limit of a sequence to prove that

   (a) $\displaystyle\lim_{n\to\infty} \frac{n+1}{n+2} = 1$;

   (b) $\displaystyle\lim_{n\to\infty} \frac{n^3-1}{n^4-1} = 0$;

   (c) $\displaystyle\lim_{n\to\infty} \frac{n!}{n^n} = 0$;

3. Use the formal definition to prove that the following sequences a_n diverge as $n \to \infty$.

   (a) $a_n = 1 + nd$, $d \neq 0$.

   (b) $a_1 = 1$, $a_{n+1} = 2^{a_n}$ for $n \in \mathbb{N}$.

4. (a) Suppose that $\{a_n\}$ is a convergent sequence for which $0 \leq a_n \leq 1$ for all $n \in \mathbb{N}$ and $L = \lim_{n\to\infty} a_n$. Prove that $L \in [0,1]$.

   (b) Find a convergent sequence $\{a_n\}$ of points all in $(0,1)$ such that $\lim_{n\to\infty} a_n$ is not in $(0,1)$.

5. Show that the formal definition of convergence of a sequence that we gave in class is equivalent to the following slight modification:

   We write $\displaystyle\lim_{n\to\infty} s_n = L$ provided that for every positive integer m there is an integer N so that $|s_n - L| < \frac{1}{m}$ whenever $n \geq N$.