Mathematics 3A03 Real Analysis I
2019 ASSIGNMENT 1 (Solutions)

This assignment was **due in the appropriate locker** on **Friday 18 Jan 2019 at 1:25pm**.

1. Prove that $\sqrt{13}$ is irrational.

Solution: Suppose, in order to derive a contradiction, that $\sqrt{13} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m,n) = 1$ such that $m/n = \sqrt{13}$.

$$\therefore \qquad \left(\frac{m}{n}\right)^2 = \left(\sqrt{13}\right)^2 \qquad \Longrightarrow \qquad \frac{m^2}{n^2} = 13 \qquad \Longrightarrow \qquad m^2 = 13n^2.$$

Thus, m^2 is a multiple of 13. It follows—from analysis we will give below—that m is a multiple of 13. Therefore, $m = 13k$ for some $k \in \mathbb{N}$, which implies $m^2 = 169k^2 = 13n^2$, and hence $13k^2 = n^2$. Thus, n^2 is a multiple of 13, which implies—again from our analysis below—that n in a multiple of 13.

Thus, both m and n contain a factor of 13, which contradicts $\gcd(m,n) = 1$. Our initial assumption that $\sqrt{13} \in \mathbb{Q}$ must therefore be false, and we can conclude that $\sqrt{13} \notin \mathbb{Q}$. However, this argument depends on the fact that:

> A natural number m contains a factor of 13 *if and only if* its square m^2 contains a factor of 13.

This a straightforward consequence of the Fundamental Theorem of Arithmetic, but rather than pulling that non-trivial result out of the air, we will prove the above statement directly. We need to prove both the "if" and "only if" directions of this statement. The "only if" direction is easier.

"Only if" direction: If m is a multiple of 13 then there is another integer k such that $m = 13k$, which implies that $m^2 = 169k^2 = 13(13k^2)$, *i.e.*, m^2 is also a multiple of 13.

"If" direction: If m is not a multiple of 13 then there must be integers k and ℓ such that $k \geq 0$, $0 < \ell < 13$ and $m = 13k + \ell$. Consequently, $m^2 = (13k + \ell)^2 = 169k^2 + 26k\ell + \ell^2 = 13(13k^2 + 2k\ell) + \ell^2$. Now for each of the 12 possible values of $\ell \in \{1, 2, \ldots, 12\}$ we can easily verify that ℓ^2 is not divisible by 13. Hence m^2 does not contain a factor of 13.

Thus, m is a multiple of 13 if and only if m^2 is a multiple of 13. $\square$

2. (a) Prove that if $0 < a < b$ then

$$a < \sqrt{ab} < \frac{a+b}{2} < b\,. \tag{1a}$$

Solution: First note that if $0 < x < y$ then from order axiom O4 with $z = x$, it follows that

$$0 < x < y \quad \Longrightarrow \quad x^2 < xy\,, \tag{$*$}$$

whereas with $z = y$ the same axiom implies

$$0 < x < y \quad \Longrightarrow \quad xy < y^2\,, \tag{$**$}$$

and hence from transitivity (axiom O2) we have

$$0 < x < y \quad \Longrightarrow \quad x^2 < y^2\,. \tag{$*^3$}$$

We can also reverse the implications: Suppose $x > 0$ and $y > 0$, and we know $x^2 < y^2$. Then it follows that $x < y$. To see this, note that if $x > y$ then ($*^3$) implies $x^2 > y^2$, which is false, and if $x = y$ then $x^2 = y^2$, which is false. Therefore, from trichotomy (O1) we must have $x < y$. In summary,

$$\text{if} \qquad x > 0 \text{ and } y > 0 \qquad \text{then} \qquad x < y \iff x^2 < y^2\,. \tag{$\heartsuit$}$$

Now, since $a < b$ and $a > 0$, order axiom O4 implies

$$a^2 < ab\,,$$

from which ($\heartsuit$) implies

$$a < \sqrt{ab}\,,$$

which establishes the first of the three inequalities in (1a). Next, since $a < b$, axiom O3 implies $a + b < b + b = 2b$, hence

$$\frac{a+b}{2} < b\,,$$

establishing the third inequality in (1a). Finally, to establish the middle inequality in (1a), note that if $a > 0$ and $b > 0$ then

$$\begin{aligned}
\sqrt{ab} < \frac{a+b}{2} &\iff 2\sqrt{ab} < a + b \\
&\iff 4ab < (a+b)^2 && \text{from } (\heartsuit) \\
&\iff 4ab < a^2 + 2ab + b^2 \\
&\iff 0 < a^2 - 2ab + b^2 \\
&\iff 0 < (a-b)^2\,.
\end{aligned}$$

Since the last inequality is true for any $a, b \in \mathbb{R}$, and all the steps are reversible for $a, b > 0$, we have the middle inequality in (1a) as required. □

Note: The approach taken to prove the middle inequality is often useful, *i.e.,* if you don't see a direct approach to prove something then manipulate what you are trying to prove to try to get it into a form that is easy to see is true. Provided all your steps are reversible, you are done.

(b) Prove that for any $a, b \geq 0$,

$$\sqrt{ab} \leq \frac{a+b}{2}\,. \tag{1b}$$

Note: This is a special case of the ***arithmetic-geometric mean inequality***.

Solution: In part (a), we proved that if $a > 0$ and $b > 0$ then $\frac{a+b}{2} > \sqrt{ab}$. In fact, each step of the argument is valid if we replace $>$ with $\geq$ everywhere. □

3. The maximum of two numbers x and y is denoted by $\max(x, y)$. Thus $\max(-1, 3) = \max(3, 3) = 3$ and $\max(-1, -4) = \max(-4, -1) = -1$. The minimum of x and y is denoted by $\min(x, y)$. Prove that

$$\max(x, y) = \frac{x + y + |y - x|}{2}\,, \tag{2a}$$

$$\min(x, y) = \frac{x + y - |y - x|}{2}\,. \tag{2b}$$

Derive a formula for $\max(x, y, z)$ and $\min(x, y, z)$, using, for example

$$\max(x, y, z) = \max(x, \max(y, z))\,. \tag{3}$$

Solution: If $x \leq y$ then

$$|y - x| = y - x \qquad \Longrightarrow \qquad \frac{x + y + |y - x|}{2} = \frac{x + y + y - x}{2} = \frac{2y}{2} = \max(x, y)\,,$$

and a similar calculation yields the result if $x > y$.

For the "triple" formulae, we have

$$\begin{aligned}
\max(x, y, z) &= \max(x, \max(y, z)) \\
&= \frac{x + \max(y, z) + |\max(y, z) - x|}{2} \\
&= \frac{x + \frac{y+z+|z-y|}{2} + \left|\frac{y+z+|z-y|}{2} - x\right|}{2} \\
&= \frac{y + z + |z - y| + 2x + |y + z + |z - y| - 2x|}{4}\,.
\end{aligned}$$

and

$$\begin{aligned}
\min(x, y, z) &= \min(x, \min(y, z)) \\
&= \frac{x + \min(y, z) - |\min(y, z) - x|}{2} \\
&= \frac{x + \frac{y+z-|z-y|}{2} - \left|\frac{y+z-|z-y|}{2} - x\right|}{2} \\
&= \frac{y + z - |z - y| + 2x - |y + z - |z - y| - 2x|}{4}\,.
\end{aligned}$$

4. Given $\varepsilon > 0$, prove that if

$$|x - x_0| < \min\left(\frac{\varepsilon}{2(|y_0| + 1)}, 1\right) \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)}\,, \tag{4}$$

then $|xy - x_0 y_0| < \varepsilon$.

<u>Hint</u>: Don't try to use the formula for min that you proved in the previous problem; it is irrelevant here. Do notice that the first condition is equivalent to two inequalities (neither involving min), both of which are needed. Since the hypotheses (4) provide information only about $x - x_0$ and $y - y_0$, it might not surprise you that the proof depends upon writing $xy - x_0 y_0$ in a way that involves $x - x_0$ and $y - y_0$.

Solution: The key is to add and substract something from $xy - x_0 y_0$ so that we can make use of what we are given in the new expression. Rather than xy we want $x(y - y_0)$, so let's add and substract xy_0.

$$\begin{aligned} |xy - x_0 y_0| &= |xy - xy_0 + xy_0 - x_0 y_0| \\ &= |x(y - y_0) + (x - x_0) y_0| \end{aligned}$$

To proceed further, we need the triangle inequality,

$$|x + y| \le |x| + |y|\,, \qquad \forall x, y \in \mathbb{R}, \tag{$\spadesuit$}$$

which can be proved, for example, by considering the cases in which x and y have the same sign or the opposite sign. Exploiting this in our equation above, we have

$$\begin{aligned} |xy - x_0 y_0| &\le |x(y - y_0)| + |(x - x_0) y_0| && \text{from } (\spadesuit) \\ &< \frac{|x|\,\varepsilon}{2(|x_0| + 1)} + \frac{|y_0|\,\varepsilon}{2(|y_0| + 1)} && \text{from } (4) \end{aligned}$$

This is progress, but we still have a variable (x) on the RHS of the inequality and we need a constant. To cure that problem, we use the other inequality on the LHS of (4), *i.e.,* $|x - x_0| < 1$, which implies—via adding zero and then using the triangle inequality—that

$$|x| = |x - x_0 + x_0| \le |x - x_0| + |x_0| < 1 + |x_0|\,.$$

This allows us to replace $|x|$ with $|x_0| + 1$ in our calculation above to obtain

$$\begin{aligned} |xy - x_0 y_0| &< \frac{(|x_0| + 1)\varepsilon}{2(|x_0| + 1)} + \frac{|y_0|\,\varepsilon}{2(|y_0| + 1)} \\ &= \frac{\varepsilon}{2} + \frac{|y_0|\,\varepsilon}{2(|y_0| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon\,, \end{aligned}$$

as required. □

5. For each of the following sets, find the greatest lower bound (inf), least upper bound (sup), minimum (min) and maximum (max), if they exist. If any of these do not exist, then indicate accordingly. Justify your assertions.

   (a) $(-7, \infty)$.

   (b) $\{\frac{1}{x} : x \in \mathbb{N} \text{ and } x \text{ is prime}\}$.

   (c) $\{(a+b)^n : a, b \in \mathbb{R},\ -\frac{1}{2} < a < b < \frac{1}{2},\ n \in \mathbb{N}\}$.

   Solution: The answers to the questions are most easily summarized in a table:

	Set	inf	sup	min	max
(a)	$(-7, \infty)$	-7	$\nexists$	$\nexists$	$\nexists$
(b)	$\{\frac{1}{x} : x \in \mathbb{N} \text{ and } x \text{ is prime}\}$	0	$\frac{1}{2}$	$\nexists$	$\frac{1}{2}$
(c)	$\{(a+b)^n : a, b \in \mathbb{R},\ -\frac{1}{2} < a < b < \frac{1}{2},\ n \in \mathbb{N}\}$	-1	1	$\nexists$	$\nexists$

   To justify the entries in this table, consider the following:

   (a) This interval is bounded below, with greatest lower bound -7, and not bounded above. Since the interval is open, it has no minimum.

   (b) Primes are positive numbers and the smallest prime is 2. Hence the largest element of this set is $\frac{1}{2}$. On the other hand, there is no largest prime[1], so there are elements of this set that are arbitrarily close to zero, from which it follows that the greatest lower bound is zero.

   (c) $-\frac{1}{2} < a < b < \frac{1}{2}$ implies $-1 < a + b < 1$, which in turn implies $-1 < (a+b)^n < 1$ for all $n \in \mathbb{N}$. Moreover, we can choose a and b such that their sum is as close as we like to -1 or 1.

[1] If you are not familiar with a proof, consider Euclid's argument: Suppose there are finitely many primes, say $p_1, p_2, \ldots, p_n$. Let $x = (p_1 p_2 \cdots p_n) + 1$. x is an integer, but cannot be prime, because $x > p$ for all primes p. Yet no prime is a factor of x, which implies x is prime! $\Rightarrow\Leftarrow$