Mathematics 3A03 Real Analysis I
2019 ASSIGNMENT 1

This assignment is **due in the appropriate locker** on **Friday 18 Jan 2019 at 1:25pm**.

1. Prove that $\sqrt{13}$ is irrational.

2. (a) Prove that if $0 < a < b$ then
$$a < \sqrt{ab} < \frac{a+b}{2} < b\,. \tag{1a}$$
   (b) Prove that for any $a, b \geq 0$,
$$\sqrt{ab} \leq \frac{a+b}{2}\,. \tag{1b}$$
   Note: This is a special case of the ***arithmetic-geometric mean inequality***.

3. The maximum of two numbers x and y is denoted by $\max(x, y)$. Thus $\max(-1, 3) = \max(3, 3) = 3$ and $\max(-1, -4) = \max(-4, -1) = -1$. The minimum of x and y is denoted by $\min(x, y)$. Prove that
$$\max(x, y) = \frac{x + y + |y - x|}{2}\,, \tag{2a}$$
$$\min(x, y) = \frac{x + y - |y - x|}{2}\,. \tag{2b}$$
   Derive a formula for $\max(x, y, z)$ and $\min(x, y, z)$, using, for example
$$\max(x, y, z) = \max(x, \max(y, z))\,. \tag{3}$$

4. Given $\varepsilon > 0$, prove that if
$$|x - x_0| < \min\left(\frac{\varepsilon}{2(|y_0| + 1)}, 1\right) \quad \text{and} \quad |y - y_0| < \frac{\varepsilon}{2(|x_0| + 1)}\,, \tag{4}$$
   then $|xy - x_0y_0| < \varepsilon$.

   Hint: Don't try to use the formula for min that you proved in the previous problem; it is irrelevant here. Do notice that the first condition is equivalent to two inequalities (neither involving min), both of which are needed. Since the hypotheses (4) provide information only about $x - x_0$ and $y - y_0$, it might not surprise you that the proof depends upon writing $xy - x_0y_0$ in a way that involves $x - x_0$ and $y - y_0$.

5. For each of the following sets, find the greatest lower bound (inf), least upper bound (sup), minimum (min) and maximum (max), if they exist. If any of these do not exist, then indicate accordingly. Justify your assertions.

   (a) $(-7, \infty)$.
   (b) $\{\frac{1}{x} : x \in \mathbb{N}$ and x is prime$\}$.
   (c) $\{(a + b)^n : a, b \in \mathbb{R},\ -\frac{1}{2} < a < b < \frac{1}{2},\ n \in \mathbb{N}\}$.