Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 6 (Solutions)

This assignment was **due in the appropriate locker** on **Monday 4 Dec 2017 at 2:25pm**.

1. Use the Fundamental Theorem of Calculus and Darboux's Theorem to give another proof of the Intermediate Value Theorem.

   Solution: Let $a < b$. If f is continuous on $[a, b]$ then by FTC $F(x) = \int_a^x f$ is differentiable on $[a, b]$ and $F' = f$. Thus f is the derivative of a function (F in particular) on $[a, b]$ so by Darboux's theorem, f satisfies the intermediate value property. This proves the IVT for a closed interval $[a, b]$. Now suppose f is defined only on the interval $(a, b]$. Then the argument above technically fails because the definition of F requires f to be defined on $[a, b]$. However, if $a < c < b$ then the argument above does show that f satisfies the IVP on $[c, b]$. Since this is true for <u>any</u> $c \in (a, b)$, we have the IVP on $(a, b]$ as required. If the interval in question is open at b then we obtain the IVP by a similar argument. □

2. An ***integral equation*** is an equation involving integrals of an unknown function. A solution of an integral equation is a function f that satisfies the equation. Consider the integral equation

$$\int_0^x f = \big(f(x)\big)^2 + C\,, \qquad (*)$$

   where $C \in \mathbb{R}$ is a constant.

   (a) For $C \neq 0$, find all <u>continuous</u> solutions of (*) for which f has at most one zero.

   Solution: Equation (*) is assumed hold for all $x \in \mathbb{R}$, and hence in particular for $x = 0$. Therefore,

$$0 = \int_0^0 f = \big(f(0)\big)^2 + C \quad\implies\quad \big(f(0)\big)^2 = -C\,. \qquad (\heartsuit)$$

   Thus, $f(0) = 0 \iff C = 0$, and if $C > 0$ then there are no functions f that satisfy (*). What if $C < 0$?

   If we restrict attention to continuous solutions of (*), *i.e.*, continuous functions f, then applying FTC to (*) implies $f^2 + C$ is differentiable (hence so is f^2) and

$$f(x) = \frac{d}{dx}\int_0^x f = \frac{d}{dx}\Big[\big(f(x)\big)^2 + C\Big] = \frac{d}{dx}\Big[\big(f(x)\big)^2\Big] = 2f(x)f'(x)\,. \qquad (**)$$

   In order to write the last equality, we need to know that f is differentiable at x, not just continuous. We will return to that below, but for now notice that Equation (**) implies that if $f(x) \neq 0$ then $f'(x) = \frac{1}{2}$. We are given, moreover, that f has at most one zero, so with the exception of at most one point, we have

$$f(x) = \frac{x}{2} + K\,, \qquad (***)$$

for some constant $K \in \mathbb{R}$. In fact, since f is continuous, Equation (***) must hold for all $x \in \mathbb{R}$. In addition, inserting $x = 0$ yields

$$K = f(0) = \pm\sqrt{-C}\,.$$

It follows that if $C < 0$ then then there are exactly two continuous solutions of (*) with at most one zero, namely

$$f_+(x) = \frac{x}{2} + \sqrt{-C} \qquad \text{and} \qquad f_-(x) = \frac{x}{2} - \sqrt{-C}\,. \tag{$\spadesuit$}$$

Now let us return to the issue of differentiability of f, remembering that we used that fact to derive our two continuous solutions. Consider any particular $x_0 \in \mathbb{R}$ for which $f(x_0) \neq 0$. We have either $f(x_0) > 0$ or $f(x_0) < 0$. Suppose $f(x_0) > 0$. Then, since f is continuous, the neighbourhood sign lemma implies $f(x) > 0$ in a neighbourhood of x_0, say $(x_0 - \delta, x_0 + \delta)$. Consequently, for all $x \in (x_0 - \delta, x_0 + \delta)$ we have

$$f(x) = \sqrt{\big(f(x)\big)^2}\,.$$

Hence, on the interval $(x_0 - \delta, x_0 + \delta)$, f is the composition of two differentiable functions, namely f^2 and $\sqrt{\cdot}$ (the crucial point here is that the square root function is differentiable at points where its argument is strictly positive). Thus, f is differentiable in a neighbourhood of every $x_0 \in \mathbb{R}$, except points at which f is zero. But there is at most one such point, which is therefore contained in a neighbourhood of the form $(x_0 - \delta, x_0 + \delta)$ on which f is differentiable. Therefore f is differentiable everywhere. □

(b) For $C \neq 0$, find a solution of (*) that is 0 on an interval $(-\infty, b]$ with $b < 0$, but non-zero for $x > b$.

Solution: Given $C < 0$, we can construct such a solution using f_+ defined in ($\spadesuit$) above. Let

$$b = -2\sqrt{-C}\,,$$

and define

$$f(x) = \begin{cases} 0 & x \leq b \\ f_+(x) & b < x. \end{cases}$$

To see that this function is a solution to (*), not that we know from part (a) that it is a solution on $[b, \infty)$. In addition, note that

$$\int_0^b f = \int_0^b f_+ = \int_0^b \left(\frac{x}{2} + \sqrt{-C}\right) dx = \left(\frac{x^2}{4} + \big(\sqrt{-C}\big)x + \text{const}\right)\Bigg|_0^b$$
$$= \frac{b^2}{4} + \big(\sqrt{-C}\big)b = \frac{b^2}{4} - \frac{b^2}{2} = -\frac{b^2}{4}\,,$$

which can also be seen directly geometrically since we just have minus the area of a triangle: $-\frac{(-b)\cdot f(0)}{2} = \frac{b\cdot\sqrt{-C}}{2} = -\frac{b^2}{4}$. Now for any $x < b$, there is no further contribution to the integral on the LHS of (*) and $f(x) = f(b)$ on the RHS, so f is a solution of (*) for all $x \in \mathbb{R}$. □

(c) For $C = 0$, and any interval $[a, b]$ with $a < 0 < b$, find a solution of (*) that is 0 on $[a, b]$, but non-zero elsewhere.

Solution: Let

$$f(x) = \begin{cases} 0 & a \leq x \leq b, \\ \frac{x}{2} - \frac{a}{2} & x < a, \\ \frac{x}{2} - \frac{b}{2} & x > b. \end{cases}$$

Note that since $C = 0$, (*) is simply $\displaystyle\int_0^x f = \big(f(x)\big)^2$. The zero function is one solution to this equation on $\mathbb{R}$, and it remains a solution on $[a, b]$ for $a < 0 < b$. For $x > b$, the area bounded by the function f is a triangle with base $x - b$ and height $f(x) = (x - b)/2$, hence $\int_0^b f(x) = \frac{1}{2}(x - b)(x - b)/2 = (x - b)^2/4$; since we also have $\big(f(x)\big)^2 = \big((x - b)/2\big)^2 = (x - b)^2/4$, f satisfies (*) for any $x > b$. The analysis is similar for $x < a$, except that we need to be careful to keep track of signs. □

3. For each of the following sequences $\{f_n\}$, determine the pointwise limit of $\{f_n\}$ (if it exists) on the indicated interval, and establish whether $\{f_n\}$ converges uniformly to this function.

(i) $f_n(x) = \dfrac{e^x}{x^{2n}}$, on $(1, \infty)$.

Solution: Note that the numerator does not depend on n. Consequently, for any $x > 1$ we have the pointwise limit

$$f(x) = \lim_{n\to\infty} \frac{e^x}{x^{2n}} = e^x \lim_{n\to\infty} \frac{1}{x^{2n}} = 0\,.$$

On the other hand, for any given $n \in \mathbb{N}$ we have

$$\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \frac{e^x}{x^{2n}} = \infty\,,$$

so every f_n is unbounded. Since the limit function $f \equiv 0$ is bounded, the convergence cannot be uniform. □

(ii) $f_n(x) = e^{-nx^2}$, on $[-1, 1]$.

Solution: For all $n \in \mathbb{N}$ we have $f_n(0) = e^0 = 1$. In addition, for all $x \neq 0$ we have $\lim_{n\to\infty} f_n(x) = 0$. Therefore, the pointwise limit is

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

Since each f_n is continuous but f is not continuous, the convergence is not uniform. □

(iii) $f_n(x) = \dfrac{e^{-x^2}}{n}\,, \qquad$ on $\mathbb{R}$.

Solution: Since the numerator does not depend on n, we have $\forall x \in \mathbb{R}$,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{e^{-x^2}}{n} = e^{-x^2} \lim_{n\to\infty} \frac{1}{n} = 0\,.$$

Therefore the pointwise limit is $f \equiv 0$. In fact, $f_n \to f$ uniformly. Note that $0 < e^{-x^2} \le 1$ for all $x \in \mathbb{R}$, and hence $0 < f_n(x) \le \frac{1}{n}$ for all $x \in \mathbb{R}$. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\varepsilon$. Then for all $n \ge N$, we have $\frac{1}{n} < \varepsilon$ and hence $0 < f_n(x) < \varepsilon$ for all $x \in \mathbb{R}$, *i.e.*, $f_n(x)$ converges uniformly to the zero function. □

4. Suppose that $\{f_n\}$ is a sequence of nonnegative bounded functions on $A \subseteq \mathbb{R}$, and let $M_n = \sup f_n$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, does it follow that $\sum_{n=1}^{\infty} M_n$ converges (a converse to the Weierstrass M-test)?

Solution: No, it does not follow. Consider $A = [1, \infty)$ and define

$$f_n(x) = \begin{cases} \frac{1}{n} & n \le x < n+1 \\ 0, & \text{otherwise.} \end{cases}$$

Then each f_n is non-negative and bounded and the series converges uniformly to

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \frac{1}{\lfloor x \rfloor}\,.$$

But $M_n = \sup f_n = \frac{1}{n}$ and $\sum_{n=1}^{\infty} M_n$ diverges. Another example, which is perhaps slightly simpler, is to define $f_n(n) = \frac{1}{n}$ and $f_n(x) = 0$ for all $x \neq n$. □

5. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$$

converges uniformly on $\mathbb{R}$.

Solution: Let $f_n(x) = \frac{x}{n(1+nx^2)}$ and note that

$$f_n'(x) = \frac{1-nx^2}{n(1+nx^2)^2}\,.$$

Moreover, $\lim_{x\to-\infty} f_n(x) = \lim_{x\to\infty} f_n(x) = 0$. Hence f_n has exactly two critical points, where $1 - nx^2 = 0$, *i.e.*, $x = \pm\frac{1}{\sqrt{n}}$. Thus

$$-\frac{1}{2n^{3/2}} = f_n(-\frac{1}{\sqrt{n}}) \le f_n(x) \le f_n(\frac{1}{\sqrt{n}}) = \frac{1}{2n^{3/2}}\,, \qquad \forall x \in \mathbb{R}.$$

Consequently, if we let $M_n = \frac{1}{2n^{3/2}}$ then the series $\sum_{n=1}^{\infty} f_n(x)$ converges by the Weierstrass M-test. □