Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 6

This assignment is **due in the appropriate locker** on **Monday 4 Dec 2017 at 2:25pm**.

1. Use the Fundamental Theorem of Calculus and Darboux's Theorem to give another proof of the Intermediate Value Theorem.

2. An ***integral equation*** is an equation involving integrals of an unknown function. A solution of an integral equation is a function f that satisfies the equation. Consider the integral equation
$$\int_0^x f = \big(f(x)\big)^2 + C\,, \qquad (*)$$
where $C \in \mathbb{R}$ is a constant.

   (a) For $C \neq 0$, find all continuous solutions of (*) for which f has at most one zero.

   (b) For $C \neq 0$, find a solution of (*) that is 0 on an interval $(-\infty, b]$ with $b < 0$, but non-zero for $x > b$.

   (c) For $C = 0$, and any interval $[a, b]$ with $a < 0 < b$, find a solution of (*) that is 0 on $[a, b]$, but non-zero elsewhere.

3. For each of the following sequences $\{f_n\}$, determine the pointwise limit of $\{f_n\}$ (if it exists) on the indicated interval, and establish whether $\{f_n\}$ converges uniformly to this function.

   (i) $f_n(x) = \dfrac{e^x}{x^{2n}}\,, \qquad$ on $(1, \infty)$.

   (ii) $f_n(x) = e^{-nx^2}\,, \qquad$ on $[-1, 1]$.

   (iii) $f_n(x) = \dfrac{e^{-x^2}}{n}\,, \qquad$ on $\mathbb{R}$.

4. Suppose that $\{f_n\}$ is a sequence of nonnegative bounded functions on $A \subseteq \mathbb{R}$, and let $M_n = \sup f_n$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, does it follow that $\sum_{n=1}^{\infty} M_n$ converges (a converse to the Weierstrass M-test)?

5. Prove that the series
$$\sum_{n=1}^{\infty} \frac{x}{n(1 + nx^2)}$$
converges uniformly on $\mathbb{R}$.