Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 5 (Solutions)

This assignment was **due in the appropriate locker** on **Monday 20 Nov 2017 at 2:25pm**.

1. Suppose $A \subseteq \mathbb{R}$ is open and $f : A \to \mathbb{R}$ is a function. For $U \subseteq \mathbb{R}$, define the *inverse image under* f to be the set

$$f^{-1}(U) = \{x \in A \mid f(x) \in U\}.$$

Show that f is continuous if and only if $f^{-1}(U)$ is open for every open set $U \subset \mathbb{R}$.

Proof: First assume that f is continuous and let $U \subseteq \mathbb{R}$ be open. To show that $f^{-1}(U)$ is open, it suffices to show that all points in $f^{-1}(U)$ are interior points. For this, pick $x \in f^{-1}(U) \subseteq A$. This implies $f(x) \in U$. Since U is open, there is some $\epsilon > 0$ so that $(f(x)-\epsilon, f(x)+\epsilon) \subseteq U$. We have assumed that f is continuous, so there is some $\delta > 0$ so that if $x' \in A$ satisfies $0 < |x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$. This implies

$$(x-\delta, x+\delta) \cap A \subset f^{-1}(U). \tag{1}$$

Since A is open and $x \in A$, the point x is an interior point of A. By shrinking δ, if necessary, we may therefore assume $(x-\delta, x+\delta) \subseteq A$. Then (1) implies $(x-\delta, x+\delta) \subseteq f^{-1}(U)$, which shows x is an interior point of $f^{-1}(U)$.

Conversely, assume that $f^{-1}(U)$ is open for every open set U. To prove that f is continuous, pick a point $x \in A$ and let $\epsilon > 0$. The set $U = (f(x)-\epsilon, f(x)+\epsilon)$ is open, so $f^{-1}(U)$ is open, by our assumption. The set $f^{-1}(U)$ contains x (since $f(x) \in U = f(f^{-1}(U))$), so x is an interior point of $f^{-1}(U)$. In particular, there is some $\delta > 0$ so that

$$(x-\delta, x+\delta) \subseteq f^{-1}(U).$$

Since $U = (f(x)-\epsilon, f(x)+\epsilon)$, this implies that if $x' \in A$ satisfies $0 < |x - x'| < \delta$, then $|f(x) - f(x')| < \epsilon$. □

2. Suppose $D \subseteq \mathbb{R}$ is a compact set and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define the *image of* D *under* f to be

$$f(D) = \{f(x) \ : \ x \in D\}.$$

   (a) Show that $f(D)$ satisfies the Bolzano-Weierstrass property. Show this by directly verifying the Bolzano-Weierstrass property (e.g., you can't use part (b)).

   (b) Show that $f(D)$ satisfies the Heine-Borel property. Show this by directly verifying the Heine-Borel property (e.g., you can't use part (a)). *Hint: Use Problem 1.*

Proof of (a): Let $\{y_n\}$ be a sequence in $f(D)$. From the definition of $f(D)$, we see that, for each n, we can write $y_n = f(x_n)$ for some $x_n \in D$. The set D satisfies the Bolzano-Weierstrass property, so a subsequence $\{x_{n_k}\}$ converges to a point $x_\infty \in D$. Since f is continuous, we have

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) = f(x_\infty) \in f(D),$$

which shows that the subsequence $\{y_{n_k}\}$ converges in $f(D)$. $\square$

Proof of (b): Let $\{U_\alpha\}$ be an open cover of $f(D)$. Since f is continuous, it follows from Problem 1 that $f^{-1}(U_\alpha)$ is open for each α. Moreover, the collection $\{f^{-1}(U_\alpha)\}$ is an open *cover* of D: If $x \in D$, then $f(x) \in f(D)$ and so $f(x) \in U_\alpha$ for some α, which implies that $x \in f^{-1}(U_\alpha)$. Since D is compact, the open cover $\{f^{-1}(U_\alpha)\}$ has a finite subcover, which we can denote by

$$\left\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_N})\right\}.$$

We will show that $\{U_{\alpha_1}, \ldots, U_{\alpha_N}\}$ covers $f(D)$. For this, let $y \in f(D)$. Then $y = f(x)$ for some $x \in D$, so $x \in f^{-1}(U_{\alpha_k})$ for some $1 \le k \le N$. Then $y = f(x) \in U_{\alpha_k}$. $\square$

3. (a) Give an example of a function that is bounded, but does not achieve a maximum.

   (b) Consider the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$. Show that this function is continuous, but does not satisfy the Intermediate Value Property. Why does this not contradict the Intermediate Value Theorem?

   (c) Suppose you are driving from Hamilton to Toronto, a total distance of 61 km. The legal speed limit is 100 km per hour everywhere. If it takes you 28 minutes to make your trip, have you broken the law? Justify your answer.

(a) The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x$ is bounded, but does not achieve a maximum. Indeed, it is bounded by 1, and $\sup(f(0, 1)) = 1$ so the maximum would have to be 1 if it existed. However, for all $x \in (0, 1)$, $f(x) \neq 1$.

(b) We saw in class that this function is continuous on its domain $D = \mathbb{R}\backslash\{0\}$ (it is a rational function). To see that it does not satisfy the Intermediate Value Property, note that $f(1) = 1$ and $f(-1) = -1$, but there is no $x \in (-1, 1) \cap D$ with $f(x) = 0$. This does not contradict the Intermediate Value Theorem because the theorem only applies to functions defined on an interval, and the domain of f is not an interval.

(c) It follows from the Mean Value Theorem that you broke the law. To see this, let $d(t)$ denote your distance in kilometers from Hamilton after t minutes. View this as a function $d : [0, 28] \to \mathbb{R}$, so

$$d(0) = 0, \qquad d(28) = 61.$$

It is physically reasonable to assume that d is continuous (people generally cannot teleport), and differentiable on $(0, 28)$ (you would have be a pretty crazy driver to not have a well-defined speed at each point). Then the Mean Value Theorem tells us that there is some point $c \in (0, 28)$ so that

$$d'(c) = \frac{d(28) - d(0)}{28 - 0} = \frac{61}{28}.$$

That is, at time c your speed is 61/28 kilometers per minute, which is approximately 130.71 kilometers per hour. That is, you were speeding at time c.

4. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and is such that $f'(x) \neq 0$ for all $x \in (a, b)$. Show that f is monotone.

   Proof: Pick $x_0 \in (a, b)$. Since $f'(x_0) \neq 0$, we find the following cases

   Case 1: $f'(x_0) > 0$

   Case 2: $f'(x_0) < 0$

   We will show that Case 1 implies that f is strictly increasing; the proof that Case 2 implies that f is strictly decreasing is similar (or just replace f with $-f$ to reduce Case 2 to Case 1).

   Assuming Case 1, we will first show that $f'(x) > 0$ for all $x \in (a, b)$. To see this, suppose not. Then there is some $x \in (a, b)$ with $f'(x) \leq 0$. Note that 0 is between $f'(x)$ and $f'(x_0)$. Darboux's theorem tells us that f' satisfies the Intermediate Value Property, so there is some c between x and x_0 with $f'(c) = 0$, which is a contradiction.

   Now we prove that f is increasing. For this, fix $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. Since f is differentiable on (a, b) it is continuous on $[x_1, x_2] \subset (a, b)$ and differentiable on (x_1, x_2). By the Mean Value Theorem, there is some $c \in (x_1, x_2)$ so that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

   We know that $f'(c) > 0$, and we know that $x_2 - x_1 > 0$, so this implies that $f(x_2) - f(x_1) > 0$. That is, $f(x_1) < f(x_2)$, which is what we wanted to show. $\square$

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree $n \in \mathbb{N}$. This means that there are $a_0, \ldots, a_n \in \mathbb{R}$ so that $a_n \neq 0$ and

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0, \forall x \in \mathbb{R}.$$

   (a) Show that if n is odd, then f has a zero.

   (b) Show that if n is even, then f has a maximum or a minimum.

   Proof of (a): Since $a_n \neq 0$, we have either $a_n > 0$ or $a_n < 0$. We will assume that $a_n > 0$; the other case is similar (or replace f with $-f$ to reduce the $a_n < 0$ case to the $a_n > 0$ case). Then $a_n > 0$ and the assumption that n is odd imply that

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = -\infty$$

Hence, there is some $R > 0$ so that $f(R) > 0$ and $f(-R) < 0$. By the Intermediate Value Theorem, there is some point $c \in [-R, R]$ with $f(c) = 0$. $\square$

Proof of (b): We will assume that $a_n > 0$, and we will prove that f has a minimum; the case where $a_n < 0$ is similar (f has a maximum in this case). The assumptions that $a_n > 0$ and that n is even imply

$$\lim_{x \to +\infty} f(x) = +\infty, \qquad \lim_{x \to -\infty} f(x) = +\infty.$$

This implies there is some $R > 0$ so that if $|x| \geq R$, then $f(x) \geq f(0)$. Consider the restriction of the function f to $[-R, R]$. This is a compact domain, so the Extreme Value Theorem implies that there is some $c \in (-R, R)$ so that

$$f(c) \leq f(x)$$

for all $x \in [-R, R]$. If $x \notin [-R, R]$, then the defining property of R implies that $f(x) \geq f(0)$, and since $0 \in [-R, R]$, this implies

$$f(c) \leq f(0) \leq f(x).$$

In summary, $f(c) \leq f(x)$ for all $x \in \mathbb{R}$, so f attains a minimum at c. $\square$

6. (a) Suppose $f : [a, b] \to \mathbb{R}$ is piecewise continuous. Prove that f is integrable on $[a, b]$ if and only if for all $\epsilon > 0$ there is a partition P of $[a, b]$ so that

$$U(f, P) - L(f, P) < \epsilon.$$

   (b) Suppose $f(x) = \lceil x \rceil$ for all $x \in \mathbb{R}$. Using the definition of the integral (or part (a)), prove that

$$\int_0^2 f = 3.$$

Proof of (a): Note that every piecewise continuous function is bounded; in particular, this means that $L(f, P)$ and $U(f, P)$ are well-defined for every partition P of $[a, b]$.

First assume that f is integrable on $[a, b]$, and let $\varepsilon > 0$. Then there are partitions P_u, P_ℓ of $[a, b]$ so that

$$U(f, P_u) - \int_a^b f < \epsilon/2, \qquad \int_a^b f - L(f, P_\ell) < \epsilon/2.$$

(Recall that, since f is integrable, we have $\int_a^b f = \inf\{U(f, P) : P\}$ and $\int_a^b f = \sup\{L(f, P) : P\}$ where the sets are indexed over all partitions P of $[a, b]$.) Then the union $P_u \cup P_\ell$ is another partition of $[a, b]$ that contains P_u and P_ℓ. We showed in class that this implies

$$U(f, P_u) \geq U(f, P_u \cup P_\ell) \geq L(f, P_u \cup P_\ell) \geq L(f, P_\ell).$$

Then combining these inequalities with the defining inequalities for P_u, P_ℓ gives

$$\begin{aligned} U(f, P_u \cup P_\ell) - L(f, P_u \cup P_\ell) &\leq U(f, P_u) - L(f, P_\ell) \\ &= U(f, P_u) - \int_a^b f + \int_a^b f - L(f, P_\ell) \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon, \end{aligned}$$

as desired.

Conversely, assume that, f is not integrable. Then

$$\inf \{U(f, P) \ : \ P\} > \sup \{L(f, P) \ : \ P\}$$

where the indexing is over all partitions P of $[a, b]$. Set

$$U_i = \inf \{U(f, P) \ : \ P\}, \qquad L_s = \sup \{L(f, P) \ : \ P\}$$

and

$$\epsilon = U_i - L_s$$

so $\epsilon > 0$. Then if P is any partition of $[a, b]$, we have

$$U(f, P) - U_i \geq 0, \qquad L_s - L(f, P) \geq 0$$

so

$$U(f, P) - L(f, P) \geq U_i - L_s = \epsilon.$$

□

Proof of (b): We need to show that f is integrable and that its integral over $[0, 2]$ equals 3. We will show that, for each $\epsilon > 0$, there is a partition P of $[0, 2]$ so that

$$U(f, P) = 3, \qquad L(f, P) = 3 - \epsilon. \tag{2}$$

Since f is piecewise continuous, it will follow immediately from Part (a) that f is integrable. That the value of the integral is 3 will follow from the inequalities

$$3 = U(f, P) \geq \int_0^2 f \geq L(f, P) = 3 - \epsilon,$$

and the fact that $\epsilon > 0$ was arbitrarily chosen.

To prove (2), fix $\epsilon > 0$ and consider the partition $P = \{t_0, t_1, t_2, t_3, t_4\}$ where

$$t_0 = 0, \quad t_1 = \epsilon/2, \quad t_2 = 1, \quad t_3 = 1 + \epsilon/2, \quad t_4 = 2.$$

Define M_k and m_k to be the supremum and infimum, respectively, of f on $[t_{k-1}, t_k]$. Then

$$\begin{aligned} U(f, P) &= M_1(t_1 - t_0) + M_2(t_2 - t_1) + M_3(t_3 - t_2) + M_4(t_4 - t_3) \\ &= 1(\epsilon/2) + 1(1 - \epsilon/2) + 2(\epsilon/2) + 2(1 - \epsilon/2) \\ &= 1 + 2 = 3 \end{aligned}$$

Similarly,

$$\begin{aligned} L(f,P) &= m_1(t_1-t_0)+m_2(t_2-t_1)+m_3(t_3-t_2)+m_4(t_4-t_3) \\ &= 0(\epsilon/2)+1(1-\epsilon/2)+1(\epsilon/2)+2(1-\epsilon/2) \\ &= 1-\epsilon/2+\epsilon/2+2-\epsilon \\ &= 3-\epsilon. \end{aligned}$$

□