Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 5

This assignment is **due in the appropriate locker** on **Monday 20 Nov 2017 at 2:25pm**.

1. Suppose $A \subseteq \mathbb{R}$ is open[1] and $f : A \to \mathbb{R}$ is a function. For $U \subseteq \mathbb{R}$, define the *inverse image under f* to be the set
$$f^{-1}(U) = \{x \in A \mid f(x) \in U\}\,.$$
Show that f is continuous if and only if $f^{-1}(U)$ is open for every open set $U \subset \mathbb{R}$.

2. Suppose $D \subseteq \mathbb{R}$ is a compact set and $f : \mathbb{R} \to \mathbb{R}$ is continuous. Define the *image of D under f* to be
$$f(D) = \{f(x) \;:\; x \in D\}\,.$$
   (a) Show that $f(D)$ satisfies the Bolzano-Weierstrass property. Show this by directly verifying the Bolzano-Weierstrass property (e.g., you can't use part (b)).
   (b) Show that $f(D)$ satisfies the Heine-Borel property. Show this by directly verifying the Heine-Borel property (e.g., you can't use part (a)). *Hint: Use Problem 1.*

3. (a) Give an example of a function that is bounded, but does not achieve a maximum.
   (b) Consider the function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ defined by $f(x) = 1/x$. Show that this function is continuous, but does not satisfy the Intermediate Value Property. Why does this not contradict the Intermediate Value Theorem?
   (c) Suppose you are driving from Hamilton to Toronto, a total distance of 61 km. The legal speed limit is 100 km per hour everywhere. If it takes you 28 minutes to make your trip, have you broken the law? Justify your answer.

4. Suppose $f : (a, b) \to \mathbb{R}$ is differentiable and is such that $f'(x) \neq 0$ for all $x \in (a, b)$. Show that f is monotone.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree $n \in \mathbb{N}$. This means that there are $a_0, \ldots, a_n \in \mathbb{R}$ so that $a_n \neq 0$ and
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x^1 + a_0, \qquad \forall x \in \mathbb{R}.$$
   (a) Show that if n is odd, then f has a zero.
   (b) Show that if n is even, then f has a maximum or a minimum.

[1] Alternatively, if you wish, you can consider any $A \subseteq \mathbb{R}$, but then you must reconsider what is meant by "open" in this context. For any $A \subseteq \mathbb{R}$, define $U \subseteq A$ to be ***open in A*** if $\exists V \subseteq \mathbb{R}$ that is open in the usual sense and $U = V \cap A$.

6. (a) Suppose $f : [a, b] \to \mathbb{R}$ is piecewise continuous[2]. Prove that f is integrable on $[a, b]$ if and only if for all $\epsilon > 0$ there is a partition P of $[a, b]$ so that

$$U(f, P) - L(f, P) < \epsilon.$$

(b) Let $f(x) = \lceil x \rceil$ for all $x \in \mathbb{R}$. Using the definition of the integral (or part (a)), prove that

$$\int_0^2 f = 3.$$

[2]A function is said to be ***piecewise continuous*** on an interval if the interval can be broken into a finite number of subintervals on which the function is continuous on each open subinterval (*i.e.,* the subinterval without its endpoints) and has a finite limit at the endpoints of each subinterval.