Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 4 (Solutions)

This assignment was **due in the appropriate locker** on **Monday 6 Nov 2017 at 2:25pm**.

1. (a) Suppose $f : \mathbb{N} \to \mathbb{N}$ and $f(n) = \lfloor n/3 \rfloor$, where $\lfloor x \rfloor$ denotes the nearest integer less than or equal to x. *Prove or disprove:*

   (i) f is one-to-one (injective);
   Solution: False. $f(1) = f(2) = 0$. $\square$
   Note: The co-domain of f should have been stated as $\mathbb{Z}$. The range is $\mathbb{N} \cup \{0\}$.

   (ii) f is onto (surjective);
   Solution: True. Given $m \in \mathbb{N}$, let $n = 3m$. Then $f(n) = \left\lfloor \frac{3m}{3} \right\rfloor = \lfloor m \rfloor = m$. $\square$

   (iii) f is a one-to-one correspondence (bijection).
   Solution: False, since f is not one-to-one.

   (b) Let $\mathbb{S} = \{n^2 : n \in \mathbb{N}\}$. Construct a bijection $f : \mathbb{Z} \to \mathbb{S}$ and prove that it is bijection.

   Solution: Let $f(n) = \begin{cases} (2n)^2, & n \le 0, \\ (2n-1)^2 & n > 0. \end{cases}$.

   We will use the fact that the squares of even integers are even and the squares of odd integers are odd.

   Suppose $f(n) = f(m) = k$. If k is even, then $n, m \le 0$ and we have $k = (2n)^2 = (2m)^2 \implies n^2 = m^2 \implies |n| = |m| \implies -n = -m \implies n = m$. If k is odd then $n, m > 0$ and $k = (2n+1)^2 = (2m+1)^2 \implies 2n+1 = 2m+1 \implies n = m$. So f is one-to-one.

   Now suppose $k \in \mathbb{S}$. If $k = 0$ then let $n = 0$. Otherwise, if k is even then $\exists \ell \in \mathbb{N}$ such that $k = (2\ell)^2$, so let $n = -\ell$. If k is odd then $\exists \ell \in \mathbb{N}$ such that $k = (2\ell - 1)^2$, so let $n = \ell$. In any of these cases, we fine $f(n) = k$, so f is onto. $\square$

2. Prove:

   (a) Any closed subset of a compact set is compact.

   Solution: Suppose K is compact, hence closed and bounded. Any subset of K is automatically bounded by the bounds of K. Hence any closed subset of K is both closed and bounded, *i.e.,* compact. $\square$

   (b) Any finite set is compact.

   Solution: Let $E = \{x_1, \dots, x_n\}$ be a finite set, and let $\mathcal{U}$ be an open cover of E. For each $i = 1, \dots, n$, choose $U_i \in \mathcal{U}$ such that $x_i \in U_i$. Then $\{U_i : i = 1, \dots, n\}$ is an open cover of E that is a finite subcover of $\mathcal{U}$. Thus, E is compact. $\square$

(c) The union of finitely many compact sets is compact.

Solution: Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a finite collection of compact sets. Suppose $\mathcal{U}$ is an open cover of the union $\cup_{i=1}^n K_i$. Since each K_i is compact, there is a finite subset of $\mathcal{U}$, say $\mathcal{U}_i$, that covers K_i. Consequently, the finite collection $\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ is a finite subcover of $\cup_{i=1}^n K_i$. $\square$

(d) The union of an infinite collection of compact sets is not necessarily compact.

Solution: $\cup_{i=1}^\infty [\frac{1}{n}, 1] = (0, 1]$ is not compact. $\square$

(e) The intersection of any collection of compact sets is compact.

Solution: The intersection of any collection of closed sets is closed, and the intersection of any collection of bounded sets is bounded by the bounds of any of the original sets. Hence the intersection of any collection compact sets is compact.

Note: The statement that the intersection of compact sets is compact is true in other familiar spaces, such as $\mathbb{R}^n$. However, it is not true in general. If you study topology in the future, you will encounter spaces in which intersections of compact sets are not necessarily compact.

3. In each of the following cases, give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with the specified property:

(a) f is continuous nowhere but $|f|$ is continuous everywhere;

Solution: $f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ -1 & x \notin \mathbb{Q}. \end{cases}$ $\square$

(b) f is continuous at some point $a \in \mathbb{R}$ and nowhere else;

Solution: $f(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$ $\square$

(c) f is continuous at each $n \in \mathbb{N}$ and nowhere else.

Solution: $f(x) = \begin{cases} 1 & x \le \frac{1}{2} \wedge x \in \mathbb{Q}, \\ x - n & n - \frac{1}{2} < x \le x + \frac{1}{2} \wedge x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q}. \end{cases}$ $\square$

4. (a) Prove that if f is continuous on $[a, b]$ then it can be extended to a continuous function on $\mathbb{R}$, i.e., there is a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere and satisfies $g(x) = f(x)$ for all $x \in [a, b]$.

Solution: Let $g(x) = \begin{cases} f(a) & x < a \\ f(x) & a \le x \le b. \\ f(b) & x > b. \end{cases}$

(b) Show that a continuous function on an open interval (a, b) cannot necessarily be extended to a continuous function on $\mathbb{R}$.

Solution: Consider $f(x) = 1/x$, which is continuous on the interval $(0, 1)$. Suppose $g : \mathbb{R} \to \mathbb{R}$ is continuous and $g(x) = f(x)$ for all $x \in (0, 1)$. Then $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} f(x)$ does not exist. $\Rightarrow\Leftarrow$

5. (a) Prove that if f and g are uniformly continuous and bounded on a set $E \subseteq \mathbb{R}$ then the product fg is uniformly continuous on E.

Solution: Since f and g are both bounded on E, find $M > 0$ such that $|f(x)| < M$ and $|g(x)| < M$ $\forall x \in E$. Since f and g are uniformly continuous on E, given any $\varepsilon > 0$ $\exists \delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$ then

$$\begin{aligned} |f(x) - f(y)| &< \frac{\varepsilon}{2M}\,, \\ \text{and} \quad |g(x) - g(y)| &< \frac{\varepsilon}{2M}\,, \end{aligned}$$

and hence

$$\begin{aligned} |f(x) - f(y)|\,|g(y)| &< \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2}\,, \\ \text{and} \quad |g(x) - g(y)|\,|f(x)| &< \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2}\,. \end{aligned}$$

Consequently,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &= |f(x)g(x) - f(x)g(y) - [f(y)g(y) - f(x)g(y)]| \\ &\le |f(x)g(x) - f(x)g(y)| + |f(y)g(y) - f(x)g(y)| \\ &= |g(x) - g(y)|\,|f(x)| + |f(x) - f(y)|\,|g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

i.e., fg is uniformly continuous. □

(b) Show that the conclusion in part (a) does not hold if one of the functions is not bounded.

Solution: First note that $f(x) = x$ is uniformly continuous but unbounded on $\mathbb{R}$. If we take $g(x) = x$ also, then fg is not uniformly continous. We can achieve the same effect by choosing g to be uniformly continuous but bounded, if we ensure that g has the same slope at every integer, say, so the slope of fg becomes arbitrarily large as $x \to \infty$. Taking $g(x) = \sin(2\pi x)$ would do the job. This may bother you since we haven't constructed the trigonometric functions rigorously. You can construct a jagged example using $|x|$.

(c) Suppose f is uniformly continuous on $A \subseteq \mathbb{R}$, g is uniformly continuous on $B \subseteq \mathbb{R}$ and $f(A) \subseteq B$. Prove that $g \circ f$ is uniformly continuous on A.

Solution: Given $\varepsilon > 0$, choose $\delta' > 0$ such that if $b_1, b_2 \in B$ and $|b_1 - b_2| < \delta'$ then $|g(b_1) - f(b_2)| < \varepsilon$. Next choose $\delta > 0$ such that if $a_1, a_2 \in A$ and $|a_1 - a_2| < \delta$ then $|f(a_1) - f(a_2)| < \delta'$. We then have $|g(f(a_1)) - g(f(a_2))| < \varepsilon$, as required. □