Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 4

This assignment is **due in the appropriate locker** on **Monday 6 Nov 2017 at 2:25pm**.

1. (a) Suppose $f : \mathbb{N} \to \mathbb{N}$ and $f(n) = \lfloor n/3 \rfloor$, where $\lfloor x \rfloor$ denotes the nearest integer less than or equal to x. *Prove or disprove:*
   (i) f is one-to-one (injective);
   (ii) f is onto (surjective);
   (iii) f is a one-to-one correspondence (bijection).

   (b) Let $\mathbb{S} = \{n^2 : n \in \mathbb{N}\}$. Construct a bijection $f : \mathbb{Z} \to \mathbb{S}$ and prove that it is bijection.

2. Prove:
   (a) Any closed subset of a compact set is compact.
   (b) Any finite set is compact.
   (c) The union of finitely many compact sets is compact.
   (d) The union of an infinite collection of compact sets is not necessarily compact.
   (e) The intersection of any collection of compact sets is compact.

3. In each of the following cases, give an example of a function $f : \mathbb{R} \to \mathbb{R}$ with the specified property:
   (a) f is continuous nowhere but $|f|$ is continuous everywhere;
   (b) f is continuous at some point $a \in \mathbb{R}$ and nowhere else;
   (c) f is continuous at each $n \in \mathbb{N}$ and nowhere else.

4. (a) Prove that if f is continuous on $[a, b]$ then it can be extended to a continuous function on $\mathbb{R}$, i.e., there is a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere and satisfies $g(x) = f(x)$ for all $x \in [a, b]$.

   (b) Show that a continuous function on an open interval (a, b) cannot necessarily be extended to a continuous function on $\mathbb{R}$.

5. (a) Prove that if f and g are uniformly continuous and bounded on a set $E \subseteq \mathbb{R}$ then the product fg is uniformly continuous on E.

   (b) Show that the conclusion in part (a) does not hold if one of the functions is not bounded.

   (c) Suppose f is uniformly continuous on $A \subseteq \mathbb{R}$, g is uniformly continuous on $B \subseteq \mathbb{R}$ and $f(A) \subseteq B$. Prove that $g \circ f$ is uniformly continuous on A.