Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 3 (Solutions)

This assignment was **due in the appropriate locker** on **Friday 20 Oct 2017 at 4:25pm**.

1. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $0 \le a_n \le b_n$ for all n. Consider the sequences

$$s_n = \sum_{k=1}^{n} a_k, \qquad t_n = \sum_{k=1}^{n} b_k$$

of partial sums. Show that if $\{t_n\}$ converges, then $\{s_n\}$ converges. *Hint: Use the Monotone Convergence Theorem.*

Proof: By the Monotone Convergence Theorem, to prove that $\{s_n\}$ converges, it suffices to show that this sequence is monotone and bounded. To see that it is monotone, fix $n \ge 1$. By assumption, we have $a_{n+1} \ge 0$, and so

$$s_n = \sum_{k=1}^{n} a_k \le a_{n+1} + \sum_{k=1}^{n} a_k = \sum_{k=1}^{n+1} a_k = s_{n+1}.$$

This shows that $\{s_n\}$ is non-decreasing, and hence monotone.

To see that $\{s_n\}$ is bounded, we use the assumption that $\{t_n\}$ converges. This implies that $\{t_n\}$ is bounded above, so there is some M with $t_n \le M$ for all n (in fact, we can simply take M to be the limit of $\{t_n\}$, since $\{t_n\}$ is non-decreasing). Since $a_k \le b_k$ for all k we have

$$s_n = \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} b_k = t_n$$

for all n. Combining this with $t_n \le M$, it follows that $s_n \le M$ for all n. On the other hand, $a_n \ge 0$ implies $s_n \ge 0$. This implies $|s_n| \le M$, and so $\{s_n\}$ is bounded. □

2. (a) Let $r \in \mathbb{R}$ with $r \ne 1$. Show that $\sum_{k=0}^{\ell-1} r^k = \frac{1-r^\ell}{1-r}$ for all $\ell \ge 1$. Conclude that if $0 \le r < 1$, then $\sum_{k=0}^{\ell-1} r^k \le \frac{1}{1-r}$ for all $\ell \ge 1$.

   (b) Suppose $\{x_n\}$ is a sequence satisfying $|x_n - x_{n+1}| < 2^{-n}$ for all n. Show that $\{x_n\}$ converges. *Hint: Use the Cauchy criterion. Show that* $|x_{n+\ell} - x_n| \le 2^{-n} \sum_{k=0}^{\ell-1} 2^{-k}$ *and then simplify this using (a).*

Proof of (a): We prove the identity $\sum_{k=0}^{\ell-1} r^k = \frac{1-r^\ell}{1-r}$ using induction. The base case is

$$\sum_{k=0}^{1-1} r^k = r^0 = 1 = \frac{1-r^1}{1-r}.$$

For the inductive step, assume $\sum_{k=0}^{\ell-1} r^k = \frac{1-r^\ell}{1-r}$ for some $\ell \geq 1$. Then

$$\sum_{k=0}^{\ell} r^k = r^\ell + \sum_{k=0}^{\ell-1} r^k = r^\ell + \frac{1-r^\ell}{1-r} = \frac{r^\ell(1-r)}{1-r} + \frac{1-r^\ell}{1-r} = \frac{1-r^{\ell+1}}{1-r}.$$

This finishes the inductive step, and the proof of the identity.

Now we prove the estimate $\sum_{k=0}^{\ell-1} r^k \leq \frac{1}{1-r}$ for $0 \leq r < 1$. The inequality $r \geq 0$ implies $r^\ell \geq 0$, and so $1 - r^\ell \leq 1$. Since $r < 1$, we have $1 - r > 0$ and so dividing both sides of $1 - r^\ell \leq 1$ by $1 - r$ gives $\frac{1-r^\ell}{1-r} \leq \frac{1}{1-r}$. Using the identity proved in the previous paragraph, we conclude $\sum_{k=0}^{\ell-1} r^k \leq \frac{1}{1-r}$ for all $\ell \geq 1$. □

Proof of (b): By the Cauchy criterion, it suffices to prove that the sequence $\{x_n\}$ is Cauchy. To prove this, let $\epsilon > 0$. Take N to be a natural number greater than $\log_2(2/\epsilon)$; in particular, this implies

$$2 \cdot 2^{-N} < \epsilon. \tag{1}$$

Suppose $n, m \geq N$. If $n = m$, then we have $|x_n - x_m| = 0 < \epsilon$, and so we are done. We may therefore assume $n < m$ (the case $n > m$ is treated in exactly the same way). Then we can write $m = n + \ell$ for some $\ell \geq 1$. By the triangle inequality, we have

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n+\ell}| \\ &= |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} - \ldots + x_{n+\ell-1} - x_{n+\ell}| \\ &\leq \textstyle\sum_{k=0}^{\ell-1} |x_{n+k} - x_{n+k+1}|. \end{aligned}$$

Using the assumption that $|x_j - x_{j+1}| < 2^{-j}$, we can continue this to get

$$|x_n - x_m| < \sum_{k=0}^{\ell-1} 2^{-n-k} = 2^{-n} \sum_{k=0}^{\ell-1} 2^{-k} \leq 2^{-n} \frac{1}{1-2^{-1}} = 2 \cdot 2^{-n},$$

where we used (a) in the last line. Using $n \geq N$ and (1), we therefore have

$$|x_n - x_m| \leq 2 \cdot 2^{-n} \leq 2 \cdot 2^{-N} < \epsilon.$$

□

3. (a) Suppose A and B are countable sets. Show that the union $A \cup B$ is countable. *Hint: First consider the case where A and B are disjoint ($A \cap B = \varnothing$).*

   (b) Suppose S is a countable set that is infinite. Show that for each $n \geq 2$, the n-fold Cartesian product S^n is countable.

   (c) Consider the set

   $$P(\mathbb{N}) := \{f \mid f \text{ is a function of the form } f : \mathbb{N} \to \{0, 1\}\}$$

   consisting of all functions from $\mathbb{N}$ to $\{0, 1\}$. Prove that $P(\mathbb{N})$ is uncountable.

Proof of (a): First assume that $A \cap B = \varnothing$. There are four cases to consider.

Case 1: A and B are both finite. In this case, then $A \cup B$ is finite and hence countable.

Case 2: A and B are both infinite. Then there are bijections $f_A : \mathbb{N} \to A$ and $f_B : \mathbb{N} \to B$. Define a new function $f : \mathbb{N} \to A \cup B$ by $f(n) = f_A(n/2)$ if n is even, and $f(n) = f_B((n+1)/2)$ if n is odd. Note that if $n \in \mathbb{N}$ is even (resp. odd), then $n/2 \in \mathbb{N}$ (resp. $(n+1)/2 \in \mathbb{N}$); in particular, f is well-defined. We will show that f is a bijection, from which the countability of $A \cup B$ will follow. To see f is injective, assume $f(n) = f(m)$. First assume $f(n) \in A$. Then $f(m) \in A$ and $f(m) \notin B$, since $A \cap B = \varnothing$. It follows from the definition of f that n and m are both even, and

$$f(n) = f_A(n/2) \qquad\qquad f(m) = f_A(m/2).$$

Since $f(n) = f(m)$, this implies $f_A(n/2) = f_A(m/2)$. We have assumed that f_A is injective, so $n/2 = m/2$. This implies $n = m$. Similarly, if $f(n) \in B$, then n and m are both odd and the fact that $n = m$ follows from the injectivity of f_B.

Now we will show that f is surjective. Let $x \in A \cup B$. If $x \in A$, then since f_A is surjective, there is some $n \in \mathbb{N}$ with $f_A(n) = x$. Since $f(2n) = f_A(n)$, this implies $f(2n) = x$. Similarly, if $x \in B$, then the surjectivity of f_B implies there is some $n \in \mathbb{N}$ with $f_B(n) = x$, and so $f(2n-1) = f_B(n) = x$.

Case 3: A is finite and B is infinite. Since A is finite, there is a natural number N and a bijection $f_A : \{1, \dots, N\} \to A$. Since B is infinite and countable, there is a bijection $f_B : \mathbb{N} \to B$. Define a new function $f : \mathbb{N} \to A \cup B$ by $f(n) = f_A(n)$ if $1 \le n \le N$ and $f(n) = f_B(n - N)$ if $n \ge N + 1$ (if $n \ge N + 1$, then $n - N \in \mathbb{N}$, so this is well-defined). That f is a bijection follows by an argument similar to the one given in Case 2.

Case 4: A is infinite and B is finite. This is proved as in Case 3.

This finishes the proof in the situation where $A \cap B = \varnothing$.

Finally, we address the situation where $A \cap B \neq \varnothing$. Define $B' = B \backslash A$. Any subset of a countable set is countable, so B' is countable. We have $A \cap B' = \varnothing$, so it follows from the considerations above that $A \cup B'$ is countable. On the other hand, we have

$$A \cup B = A \cup B'$$

so $A \cup B$ is countable as well. $\square$

Proof of (b): Suppose S is countable and infinite. Then there is a bijection $f : \mathbb{N} \to S$. We will use induction to prove that S^n is countable for all $n \ge 2$. For the base case, define a function $F : \mathbb{N} \times \mathbb{N} \to S \times S$ by $F(n,m) = (f(n), f(m))$. We first claim that the function F is a bijection. For injectivity, assume $F(n,m) = F(n',m')$. This implies $f(n) = f(n')$ and $f(m) = f(m')$. Since f is injective, it follows that $n = n'$ and $m = m'$, and so $(n,m) = (n',m')$. Hence F is injective. For surjectivity, let $(x,y) \in S \times S$. Since f is surjective, there are $n, m \in S$ with $f(n) = x$ and $f(m) = y$. Then $F(n,m) = (x,y)$, so F is surjective.

In class we saw that $\mathbb{N}\times\mathbb{N}$ is countable and infinite, so there is some bijection $G : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. The composition of bijections is another bijection, so $F \circ G : \mathbb{N} \to S \times S$ is a bijection. It follows that $S \times S$ is countable, so this finishes the proof of the base case.

For the inductive step, assume that S^n is countable for some $n \geq 2$. Since S is infinite, S^n is infinite as well, and so there is some bijection $g : \mathbb{N} \to S^n$. Consider the function

$$F' : \mathbb{N}\times\mathbb{N} \to S^{n+1} = S^n \times S$$

given by $F'(n,m) = (g(n), f(m))$. The same proof as above shows that F' is a bijection. Consequently, $F' \circ G : \mathbb{N} \to S^{n+1}$ is a bijection (where G is as above), and so S^{n+1} is countable. $\square$

Proof of (c): We prove this by contradiction. Suppose $P(\mathbb{N})$ is countable. Then there is a bijection $F : \mathbb{N} \to P(\mathbb{N})$. We will show that F is not surjective, which will be a contradiction. Note that for each $n \in \mathbb{N}$, the value $F(n)$ is in $P(\mathbb{N})$ and so is a function from $\mathbb{N}$ to $\{0,1\}$. Let $f_n = F(n)$ for each n. So, for each $n \in \mathbb{N}$, $f_n : \mathbb{N} \to \{0,1\}$, *i.e.*, $f_n(k)$ is either 0 or 1 for each $k \in \mathbb{N}$. In particular, $f_n(n)$ is either 0 or 1 for each $n \in \mathbb{N}$. Now define a function $f : \mathbb{N} \to \{0,1\}$ as follows:

$$f(n) = \begin{cases} 1 & \text{if } f_n(n) = 0, \\ 0 & \text{if } f_n(n) = 1. \end{cases}$$

Then $f \in P(\mathbb{N})$. The fact that F is not surjective now follows from the next claim (remember that $F(n) = f_n$):

Claim: $f_n \neq f$ for all $n \in \mathbb{N}$.

We prove this claim by contradiction. Suppose there is some $n \in \mathbb{N}$ so that $f_n = f$. Evaluating both sides at n gives $f_n(n) = f(n)$. But we defined f so that $f_n(n) \neq f(n)$, which is a contradiction. This proves the claim, and therefore finishes the proof of (c). $\square$

Remark: Part (c) is really just a more abstract presentation of Cantor's diagonal argument for the uncountability of $\mathbb{R}$.

4. Let $E = \left\{x \in \mathbb{Q} \mid -\sqrt{2} < x < 0\right\}$.
   (a) Find the closure of E in $\mathbb{R}$.
   (b) Is E closed?
   (c) Find the interior of E in $\mathbb{R}$.
   (d) Is E open?
   (e) (Bolzano-Weierstrass Property) Does every sequence of points in E have a subsequence that converges to a point in E? If so, prove it. Otherwise, construct a sequence with no subsequence converging in E.
   (f) (Heine-Borel Property) Does every open cover of E have a finite subcover? If so, prove it. Otherwise, construct an open cover that has no finite subcover.

Solutions:

(a) The closure of E is $\left[-\sqrt{2},0\right]$. So see this, let $x \in \left[-\sqrt{2},0\right]$. We need to show that x is the limit of a sequence of points in E. By the density of the rationals, for each $n \in \mathbb{N}$, there is some rational number x_n so that $|x - x_n| < 1/n$. Since $-\sqrt{2} \le x \le 0$, we may further assume that $-\sqrt{2} < x_n < 0$ (since $\forall n \in \mathbb{N}$, $0 < \frac{1}{n} < \sqrt{2} \notin \mathbb{Q}$, and $x_n \in \mathbb{Q}$). Thus $x_n \in E$ for all n and $\lim_{n\to\infty} x_n = x$.

(b) Since the closure of E is not equal to E, it follows that E is not closed.

(c) The interior of E is empty. We prove this by contradiction. If the interior of E were not empty, then there would be some $x \in E$ and $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subset E$. The irrational numbers are dense, so there is some irrational $y \in (x-\varepsilon, x+\varepsilon)$. Hence $y \in E$ and so y is rational. This contradicts the irrationality of y.

(d) Since the interior of E is not equal to E, it follows that E is not open.

(e) No, there are sequences in E that have no subsequences converging in E. To see this, consider the sequence defined by $x_n = -1/n$. Then $x_n \in E$ for all n. On the other hand, $\{x_n\}$ converges to 0, and so every subsequence of $\{x_n\}$ converges to 0 as well. Since $0 \notin E$, this finishes the argument.

(f) No, there are open covers of E that have no finite subcovers. For example, consider the collection

$$\begin{aligned}\mathcal{U} \;=\; \big\{&(-\sqrt{2}+1,0),(-\sqrt{2}+1/2,-\sqrt{2}+1),(-\sqrt{2}+1/3,-\sqrt{2}+1/2),\\ &(-\sqrt{2}+1/4,-\sqrt{2}+1/3),(-\sqrt{2}+1/5,-\sqrt{2}+1/4)\ldots\big\}\end{aligned}$$

The union of all the sets in $\mathcal{U}$ is

$$\bigcup_{n=1}^{\infty}\left(-\sqrt{2}+\frac{1}{n+1},-\sqrt{2}+\frac{1}{n}\right) \quad\cup\quad (-\sqrt{2}+1,0)\,,$$

which covers everything in the interval $(-\sqrt{2},0)$ except for the irrational numbers of the form $-\sqrt{2}+\frac{1}{n}$; in particular, $\mathcal{U}$ is a cover of E. Distinct sets in $\mathcal{U}$ have disjoint intersection, yet every set in $\mathcal{U}$ contains a point in E, so any subcover (i.e., any cover of E consisting of intervals in $\mathcal{U}$) is necessarily all of $\mathcal{U}$. Since $\mathcal{U}$ is infinite, this implies every subcover of $\mathcal{U}$ is infinite.