Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 3

This assignment is **due in the appropriate locker** on **Friday 20 Oct 2017 at 4:25pm**.

1. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences with $0 \le a_n \le b_n$ for all n. Consider the sequences
$$s_n = \sum_{k=1}^{n} a_k, \qquad t_n = \sum_{k=1}^{n} b_k$$
of partial sums. Show that if $\{t_n\}$ converges, then $\{s_n\}$ converges. *Hint: Use the Monotone Convergence Theorem.*

2. (a) Let $r \in \mathbb{R}$ with $r \neq 1$. Show that $\sum_{k=0}^{\ell-1} r^k = \frac{1-r^\ell}{1-r}$ for all $\ell \ge 1$. Conclude that if $0 \le r < 1$, then $\sum_{k=0}^{\ell-1} r^k \le \frac{1}{1-r}$ for all $\ell \ge 1$.
   (b) Suppose $\{x_n\}$ is a sequence satisfying $|x_n - x_{n+1}| < 2^{-n}$ for all n. Show that $\{x_n\}$ converges. *Hint: Use the Cauchy criterion. Show that* $|x_{n+\ell} - x_n| \le 2^{-n} \sum_{k=0}^{\ell-1} 2^{-k}$ *and then simplify this using (a).*

3. (a) Suppose A and B are countable sets. Show that the union $A \cup B$ is countable. *Hint: First consider the case where A and B are disjoint ($A \cap B = \varnothing$).*
   (b) Suppose S is a countable set. Show that for each $n \ge 2$, the n-fold Cartesian product S^n is countable.
   (c) Consider the set
$$P(\mathbb{N}) := \{f \mid f \text{ is a function of the form } f : \mathbb{N} \to \{0,1\}\}$$
consisting of all functions from $\mathbb{N}$ to $\{0,1\}$. Prove that $P(\mathbb{N})$ is uncountable.

4. Let $E = \left\{x \in \mathbb{Q} \mid -\sqrt{2} < x < 0\right\}$.
   (a) Find the closure of E in $\mathbb{R}$.
   (b) Is E closed?
   (c) Find the interior of E in $\mathbb{R}$.
   (d) Is E open?
   (e) (Bolzano-Weierstrass Property) Does every sequence of points in E have a subsequence that converges to a point in E? If so, prove it. Otherwise, construct a sequence with no subsequence converging in E.
   (f) (Heine-Borel Property) Does every open cover of E have a finite subcover? If so, prove it. Otherwise, construct an open cover that has no finite subcover.