Mathematics 3A01 Real Analysis I
2017 ASSIGNMENT 2

This assignment is **due in the appropriate locker** on **Friday 29 Sep 2017 at 4:25pm**.

1. Use the principle of mathematical induction to prove that for any $n \in \mathbb{N}$,

   (a) $\displaystyle\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$;

   Solution: We first verify the base case ($n = 1$): $\sum_{k=0}^{1} 2^k = 1 + 2 = 2^{1+1} - 1$. Next we assume the proposition is true for some $n \geq 1$ and prove that it is true for $n + 1$:

   $$\sum_{k=0}^{n+1} 2^k = \left(\sum_{k=0}^{n} 2^k\right) + 2^{n+1} = \underbrace{(2^{n+1} - 1)}_{\substack{\text{By induction}\\ \text{hypothesis}}} + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1,$$

   as required. □

   (b) if the integers $x_1, \ldots, x_n$ are odd then their product $x_1 x_2 \cdots x_n$ is odd.
   Hint: Start with $n = 2$.

   Solution: We first check the base case ($n = 2$): Suppose x and y are odd integers, *i.e.*, $\exists k, \ell \in \mathbb{Z}$ such that $x = 2k + 1$ and $y = 2\ell + 1$. Then $xy = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$, which is odd.

   Now suppose that the product of any n odd integers is odd. Consider $n + 1$ odd integers, $x_1, x_2, \ldots, x_n, x_{n+1}$. By our induction hypothesis, the product $x_* = x_1 x_2 \cdots x_n$ is odd and, therefore, by our analysis of the base case, we have $x_* x_{n+1} = x_1 x_2 \cdots x_n x_{n+1}$ is odd. □

2. Use the formal definition of a limit of a sequence to prove that

   (a) $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n^3} = 0$;

   Solution: Given $\varepsilon > 0$ we must find $N \in \mathbb{N}$ such that $|(-1)^n/n^3| < \varepsilon$ for all $n \geq N$. Note that

   $$\left|\frac{(-1)^n}{n^3}\right| < \varepsilon \iff \frac{1}{n^3} < \varepsilon \iff n^3 > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon^{1/3}}\,.$$

   We've now discovered what we need to construct a formal proof:

   Proof. Given $\varepsilon > 0$, let $N = \lceil \varepsilon^{-1/3} \rceil$. Then

   $$n \geq N \implies n \geq \lceil \varepsilon^{-1/3} \rceil \geq \varepsilon^{-1/3} \implies n^3 \geq \frac{1}{\varepsilon} \implies \frac{1}{n^3} < \varepsilon \implies \left|\frac{(-1)^n}{n^3}\right| < \varepsilon$$

   as required □

(b) $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$.

Solution: Given $\varepsilon > 0$ we must find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{n^2-1}{n^2+1} - 1\right| < \varepsilon \,.$$

Note that for any $n \in \mathbb{N}$,

$$\frac{n^2-1}{n^2+1} - 1 = \frac{n^2-1+1-1}{n^2+1} - 1 = \frac{n^2+1}{n^2+1} - \frac{2}{n^2+1} - 1 = -\frac{2}{n^2+1}\,,$$

so it is equivalent to show $2/(n^2+1) < \varepsilon$, *i.e.*, $n > \sqrt{(2/\varepsilon)-1}$ (provided $\varepsilon < 2$, which we can assume without loss of generality). Now we're ready to write our formal proof:

Proof. Let $\varepsilon > 0$. First consider the case where $\varepsilon < 2$. In this case, set $N = \left\lceil \sqrt{(2/\varepsilon)-1} \right\rceil + 1$. Then

$$\begin{aligned} n \geq N &\implies n > \left\lceil \sqrt{(2/\varepsilon)-1} \right\rceil \geq \sqrt{(2/\varepsilon)-1} \implies n^2+1 > \frac{2}{\varepsilon} \implies \frac{2}{n^2+1} < \varepsilon \\ &\implies \left|-\frac{2}{n^2+1}\right| < \varepsilon \implies \left|\frac{n^2+1}{n^2+1} - \frac{2}{n^2+1} - 1\right| < \varepsilon \\ &\implies \left|\frac{n^2-1}{n^2+1} - 1\right| < \varepsilon\,, \end{aligned}$$

as required.

Now assume $\varepsilon \geq 2$. By the above, there is some $N \in \mathbb{N}$ so that

$$\left|\frac{n^2-1}{n^2+1} - 1\right| < 1$$

for all $n \geq N$. Since $1 \leq \varepsilon$, this finishes the proof in this case as well. □

3. Use the formal definition to prove that the following sequences $\{s_n\}$ *diverge* as $n \to \infty$.

(a) $s_n = (-r)^n$ (for any $r \geq 1$) ;

Solution: Suppose $L \in \mathbb{R}$. If $L \leq 0$ then note that for all $n \in \mathbb{N}$, $|s_{2n} - L| = |r^{2n} - L| = |r^{2n} + |L|| = r^{2n} + |L| > r^{2n} \geq 1$. Similarly, if $L > 0$ then $\forall n \in \mathbb{N}$, $|s_{2n+1} - L| = |-r^{2n+1} - L| = |r^{2n+1} + L| > r^{2n+1} \geq 1$. Thus, for any $L \in \mathbb{R}$, we can find $n \in \mathbb{N}$ such that $|s_n - L| \geq 1$, implying $\{s_n\}$ does not converge to L. □

(b) $s_n = \frac{n!}{2^n}$.

Solution: Note that for $n > 4$ we have

$$\begin{aligned} s_n &= \frac{n}{2} \cdot \frac{n-1}{2} \cdots \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \\ &= \left(\frac{n}{2} \cdot \frac{n-1}{2} \cdots \frac{5}{2}\right) \cdot \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \\ &> \left(\frac{4}{2}\right)^{n-4} \frac{4}{2} \cdot \frac{3}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} \;=\; 2^{n-4}\frac{24}{16} \;=\; 2^{n-4}\frac{3}{2} \;>\; 2^{n-4} \;>\; n-4\,. \end{aligned}$$

(To be complete, we should really prove by induction that $s_n > n-4$ for all $n > 4$, but we'll leave that as an exercise.) Now, given $M > 0$, choose $N = \lceil M \rceil + 5$. Then $\forall n \geq N$, $s_n > n - 4 \geq N - 4 = \lceil M \rceil + 1 > M$, as required. $\square$

4. Suppose $s_n \to 0$ as $n \to \infty$ and that $s_n > 0$ for at least one $n \in \mathbb{N}$. Prove that the set $\{s_n\}$ has a maximum value.

   Solution: Choose $k \in \mathbb{N}$ such that $s_k > 0$. Since $\{s_n\}$ converges, it is bounded above, say by $M \geq s_k$. Since $\{s_n\}$ actually converges to 0, choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $s_n < s_k$. Now consider the finite set of real numbers $\{s_1, s_2, \ldots, s_N\}$, which has a maximum, say s_ℓ. We then have $s_\ell \geq s_k > s_n$ for all $n \geq N$, so $s_\ell \geq s_n$ for all $n \in \mathbb{N}$, *i.e.*, s_ℓ is, in fact, the maximum of the entire sequence. $\square$

5. Prove that if $\lim_{n\to\infty} s_n = L$ then $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = L$.

   Solution: Given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that $\forall n \geq N$, $|s_n - L| < \varepsilon$. If we write this out for a few terms with $n \geq N$, we have

$$\begin{aligned} -\varepsilon < s_N - L < \varepsilon \\ -\varepsilon < s_{N+1} - L < \varepsilon \\ -\varepsilon < s_{N+2} - L < \varepsilon \end{aligned}$$

   If we add these and divide by 3 we have

$$-\varepsilon < \frac{1}{3} \sum_{k=N}^{N+2} s_k - L < \varepsilon \,.$$

   If we could take $N = 1$ then we'd be done, but that's not possible in general. Instead we need to find a way to argue that the first N terms are negligible in the sum in the limit. If we make the upper limit of the sum sufficiently large, this should do the trick. Hopefully these comments are sufficient to motivate the following.

   Proof. Given $\varepsilon > 0$, choose $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $|s_n - L| < \frac{\varepsilon}{2}$, *i.e.*,

$$\forall n \geq N_1, \qquad -\frac{\varepsilon}{2} < s_n - L < \frac{\varepsilon}{2} \,.$$

   In addition, since s_n converges, it is bounded, *i.e.*, $\exists m, M \in \mathbb{R}$ such that $m < s_n < M$ for all $n \in \mathbb{N}$, *i.e.*,

$$\forall n \geq 1, \qquad m - L < s_n - L < M - L \,.$$

Now for any $n > N_1$ consider the finite sequence of inequalities

$$\begin{gathered} m - L < s_1 - L < M - L \\ m - L < s_2 - L < M - L \\ \vdots \\ m - L < s_{N_1} - L < M - L \\ -\frac{\varepsilon}{2} < s_{N_1+1} - L < \frac{\varepsilon}{2} \\ \vdots \\ -\frac{\varepsilon}{2} < s_n - L < \frac{\varepsilon}{2} \end{gathered}$$

and take their sum to obtain

$$N_1(m - L) - (n - N_1)\frac{\varepsilon}{2} < \sum_{k=1}^{n} s_k - nL < N_1(M - L) + (n - N_1)\frac{\varepsilon}{2}\,.$$

Dividing by n we have

$$\frac{N_1}{n}(m - L) - \frac{n - N_1}{n}\frac{\varepsilon}{2} < \frac{1}{n}\sum_{k=1}^{n} s_k - L < \frac{N_1}{n}(M - L) + \frac{n - N_1}{n}\frac{\varepsilon}{2}\,.$$

Since $n > N_1$, it follows that

$$\frac{N_1}{n}(m - L) - \frac{\varepsilon}{2} < \frac{1}{n}\sum_{k=1}^{n} s_k - L < \frac{N_1}{n}(M - L) + \frac{\varepsilon}{2}\,.$$

Now choose $N_2 \in \mathbb{N}$, with $N_2 > N_1$, such that

$$-\frac{\varepsilon}{2} < (m - L)\frac{N_1}{N_2} \leq 0 \leq (M - L)\frac{N_1}{N_2} < \frac{\varepsilon}{2}\,,$$

which can be done, for example, by taking

$$N_2 = \max\left\{\left\lceil\frac{2N_1(L - m)}{\varepsilon}\right\rceil, \left\lceil\frac{2N_1(M - L)}{\varepsilon}\right\rceil, N_1 + 1\right\}\,.$$

Then, for all $n \geq N_2$, we have

$$-\frac{\varepsilon}{2} - \frac{\varepsilon}{2} < \frac{1}{n}\sum_{k=1}^{n} s_k - L < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\,,$$

i.e.,

$$\forall n \geq N_2, \qquad \left|\frac{1}{n}\sum_{k=1}^{n} s_k - L\right| < \varepsilon\,,$$

as required. □