Mathematics 3A03 Real Analysis I
2017 ASSIGNMENT 1 (Solutions)

This assignment was **due in the appropriate locker** on **Friday 15 Sep 2017 at 4:25pm**.

1. Prove that $-2\sqrt{2}+3$ is irrational.

   Proof: We will prove this by contradiction. If $-2\sqrt{2}+3$ were rational, then we could write
   $$-2\sqrt{2}+3=\frac{m}{n}$$
   for some integers m,n with $n\neq 0$. This implies
   $$-2\sqrt{2}=\frac{m-3n}{n}$$
   and so
   $$\sqrt{2}=\frac{3n-m}{2n}. \tag{1}$$
   We know that the difference and product of integers is again an integer. Hence $3n-m$ and $2n$ are both in $\mathbb{Z}$. Moreover, $n\neq 0$ implies $2n\neq 0$. It follows that $\frac{3n-m}{2n}\in\mathbb{Q}$ is rational. Then Equation (**??**) implies that $\sqrt{2}$ is rational. But we saw in class that $\sqrt{2}$ is not rational, so this is a contradiction. $\square$

2. Define $\mathbb{Z}_4$ to consist of the set $\{0,1,2,3\}$, together with addition $+$ and multiplication $\cdot$ defined by the following two tables.

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

   Is $\mathbb{Z}_4$ a field? Justify your answer.

   No, $\mathbb{Z}_4$ is not a field since it fails the multiplicative inverses axiom (Axiom M4 from the slides). More specifically, the element 2 has no multiplicative inverse in $\mathbb{Z}_4$. Before proving this, we need to first figure out what the multiplicative identity is, since the multiplicative identity shows up in the statement of Axiom M4. A quick look at the "1 column" of the multiplication table shows that $1x = x$ for all $x \in \mathbb{Z}_4$. This is precisely what it means for 1 to be the multiplicative identity (see Axiom M3).

Now we will prove that 2 has no multiplicative inverse in $\mathbb{Z}_4$; that is, we will show that $2y \neq 1$ for all $y \in \mathbb{Z}_4$ (this is where we use the fact that 1 is the multiplicative identity). We see from the "2 column" of the multiplication table that if $y \in \mathbb{Z}_4$, then $2y = 0$ or $2y = 2$. Since $0 \neq 1$ and $2 \neq 1$, we deduce that $2y \neq 1$ for all $y \in \mathbb{Z}_4$, as desired.

(As a side note, Axiom M4 is the only field axiom that $\mathbb{Z}_4$ fails to satisfy.)

3. For each of the following sets, find the greatest lower bound (inf), least upper bound (sup), minimum (min) and maximum (max), if they exist. If any of these do not exist, then indicate accordingly. Justify your assertions.

   (a) $(-\infty, 2]$.

   (b) $\{x : x \in \mathbb{R} \text{ and } |x| < 3\}$.

   (c) $\{\frac{n+1}{n} : n \in \mathbb{N}\}$.

   (a) The infimum of $(-\infty, 2]$ does not exist, since there is no lower bound (that said, some people write $\inf(-\infty, 2] = -\infty$ as a shorthand to indicate that there is no lower bound, so $-\infty$ is also an acceptable answer). To prove that there is no lower bound, we need to show that, for every $m \in \mathbb{R}$, there is some $x \in (-\infty, 2]$ such that $x < m$; indeed, given m, define x by setting $x = m - 1$.

   Since the infimum does not exist, the minimum does not exist either.[1] The maximum is 2 since $2 \in (-\infty, 2]$ and $x \leq 2$ for all $x \in (-\infty, 2]$. The supremum is also 2 since, whenever the maximum exists, the supremum exists and equals the maximum.

   (b) Note that $\{x : x \in \mathbb{R} \text{ and } |x| < 3\} = (-3, 3)$. First we will show that

$$\sup(-3, 3) = 3. \tag{2}$$

   To see this, note that $x < 3$ for all $x \in (-3, 3)$. Thus, the non-empty set $(-3, 3)$ is bounded above by 3. It follows that the supremum exists and $\sup(-3, 3) \leq 3$. To finish the proof of (**??**), we need to show that 3 is the *least* upper bound. Toward this end, assume x is another upper bound for $(-3, 3)$. We will have established (**??**) if we can show $3 \leq x$. Working by contradiction, assume $3 > x$. Note that the interval $(0, 3 - x)$ is nonempty, and so the intersection $(0, 3 - x) \cap (0, 3)$ is nonempty as well. Let $\epsilon \in (0, 3 - x) \cap (0, 3)$ be any number in this intersection. Since $\epsilon \in (0, 3 - x)$, it follows that $\epsilon < 3 - x$ and hence

$$3 - \epsilon > x. \tag{3}$$

   On the other hand, since $\epsilon \in (0, 3)$, we have $3 - \epsilon \in (-3, 3)$. We have assumed that x is an upper bound for $(-3, 3)$. This implies $3 - \epsilon \leq x$, which is a contradiction to (**??**).

[1] However, $-\infty$ is *not* an acceptable answer for the minimum. The property motivating this convention is that we always require that the minimum is contained in the set being considered.

Our initial assumption that $3 > x$ must be false, so we can conclude that 3 is, in fact, the supremum of $(-3, 3)$.

The set $(-3, 3)$ has no maximum. Indeed, if a maximum exists, then it would have to be the supremum, which is 3, but $3 \notin (-3, 3)$.

Similar arguments show that the inf is -3 and the min does not exist.

(c) Write $\frac{n+1}{n} = 1 + \frac{1}{n}$ for some $n \in \mathbb{N}$. Note that $n \geq 1$, so we have $0 < \frac{1}{n} \leq 1$, which gives

$$1 < 1 + \frac{1}{n} \leq 2.$$

It follows that the supremum and infimum exist and

$$1 \leq \inf\left\{\frac{n+1}{n} : n \in \mathbb{N}\right\} \leq \sup\left\{\frac{n+1}{n} : n \in \mathbb{N}\right\} \leq 2.$$

Note that if $n = 1$, then $1 + \frac{1}{n} = 2$. It follows that the value 2 is obtained in the set, so 2 is the maximum. Consequently, 2 must also be the supremum:

$$\max\left\{\frac{n+1}{n} : n \in \mathbb{N}\right\} = \sup\left\{\frac{n+1}{n} : n \in \mathbb{N}\right\} = 2.$$

Next we will show that the infimum is 1. To see this, suppose x is a lower bound for the set; we need to show that $1 \geq x$. If $1 < x$, then $0 < x - 1$, so by the Archimedean property, we can find some natural number n with $\frac{1}{n} < x - 1$. Then

$$\frac{n+1}{n} = 1 + \frac{1}{n} < x.$$

This contradicts the assumption that x is a lower bound, so $1 \geq x$ as desired.

Finally, the minimum does not exist, since if it did it would have to be 1 (the greatest lower bound), but $1 \notin \{\frac{n+1}{n} : n \in \mathbb{N}\}$.

4. Suppose $S, T \subseteq \mathbb{R}$ are bounded, nonempty sets with $S \subseteq T$. Find relations between $\sup S, \sup T, \inf S$ and $\inf T$. Justify your assertions.

Since both sets are bounded and nonempty, their suprema and infima exist. We claim that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

Note that we automatically have $\inf S \leq \sup S$, so to prove the claim, it suffices to show $\inf T \leq \inf S$ and $\sup S \leq \sup T$. We will prove the former; the latter is similar. To see that $\inf T \leq \inf S$, note that since $\inf T$ is a lower bound for T, we have $x \geq \inf T$ for all $x \in T$. Since $S \subseteq T$, this implies

$$x \geq \inf T, \forall x \in S.$$

It follows that $\inf T$ is a lower bound for S. Since $\inf S$ is the *greatest* lower bound, we immediately conclude

$$\inf T \leq \inf S,$$

as desired.

5. Let $x, y \in \mathbb{R}$. Prove that $x = y$ if and only if $|x - y| < \epsilon$ for every $\epsilon > 0$.

   Proof: First suppose $x = y$. It follows that $|x - y| = 0$. Then if $\epsilon > 0$, we have

   $$|x - y| = 0 < \epsilon$$

   as desired.

   Now we will prove the converse. Note that it is equivalent to prove that if $x \neq y$, then there is some $\epsilon > 0$ with $|x - y| \geq \epsilon$. To show this, suppose $x \neq y$. Then $x - y \neq 0$ and so $|x - y| > 0$. Take $\varepsilon = |x - y|/2$. Then $\epsilon > 0$ and

   $$|x - y| \geq \epsilon.$$

   $\square$

6. Let $x_1, x_2 \in \mathbb{R}$ be real numbers with $x_1 < x_2$. Show that if $y \in (x_1, x_2)$, then there are rational numbers $r_1, r_2 \in \mathbb{Q}$ with $y \in (r_1, r_2)$ and $(r_1, r_2) \subseteq (x_1, x_2)$.

   Proof: Fix $y \in (x_1, x_2)$. Then $y < x_2$. Since the rational numbers are dense, there is some rational number r_2 with $y < r_2 < x_2$. Similarly, there is some rational number r_1 with $x_1 < r_1 < y$. Then $r_1 < y < r_2$ implies $y \in (r_1, r_2)$, while the inequalities $r_2 < x_2$ and $x_1 < r_1$ imply $(r_1, r_2) \subseteq (x_1, x_2)$.