Mathematics 3A03 Real Analysis I

2016 ASSIGNMENT 6

SOLUTIONS

This assignment is **due in the appropriate locker** on **Wed 7 Dec 2016 at 2:25pm**.

1. Let f be integrable on $[a,b]$, let $c \in (a,b)$, and let

$$F(x) = \int_a^x f\,, \qquad a \le x \le b.$$

For each of the following statements, give either a proof or a counterexample.

(a) If f is differentiable at c, then F is differentiable at c.

Solution: f differentiable at $c \implies f$ continuous at $c \implies F$ differentiable at c. $\square$

(b) If f is differentiable at c, then F' is continuous at c.

Solution: If f is continuous in a neighbourhood of c then F' is continuous in that neighbourhood (since $F' = f$), so in particular we would have F' continuous at c. However, we cannot assume that f is continuous anywhere except at c itself (where it is differentiable). In general, F' need not even exist anywhere but at c itself (recall the trapping principle from Assignment 5, problem 4). $\square$

(c) If f' is continuous at c, then F' is differentiable at c.

Solution: f' continuous at $c \implies f'$ exists in a neighbourhood of $c \implies f$ is continuous in a neighbourhood of $c \implies F' = f$ is continuous in that neighbourhood. Moreover, since f' exists at c, F' is differentiable at c. $\square$

2. The ***improper integral*** $\displaystyle\int_{-\infty}^{a} f$ is defined in the obvious way, as

$$\lim_{N\to-\infty} \int_N^a f\,.$$

But another kind of improper integral $\displaystyle\int_{-\infty}^{\infty} f$ is defined in an unobvious way: it is

$$\int_{-\infty}^{0} f + \int_0^{\infty} f\,,$$

provided these improper integrals both exist.

(a) Explain why $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ exists.

Solution: From the Fundamental Theorem of Calculus, we have

$$\int_0^N \frac{1}{1+x^2}\,dx = \arctan N\,, \qquad \text{for all } N \in \mathbb{R}.$$

In addition, for all $N \in \mathbb{R}$,

$$\int_{-N}^{0} \frac{1}{1+x^2}\,dx = -\int_0^{-N} \frac{1}{1+x^2}\,dx = -\arctan(-N) = \arctan N\,.$$

Now, since arctan is strictly increasing and bounded on $\mathbb{R}$, $\lim_{N\to\infty} \arctan N$ exists, so $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\lim_{N\to\infty}\arctan N$ exists (and in fact equals $2\cdot\frac{\pi}{2} = \pi$).

If the above argument makes you feel uncomfortable, that's good, because it uses facts that are perhaps familiar but were not specifically derived in this course. A better approach is to let $a_N = \int_0^N \frac{1}{1+x^2}\,dx$ for each $N \in \mathbb{N}$, and note that a_N is an increasing sequence and

$$a_N \le \sum_{n=1}^{N} \frac{1}{1+(n-1)^2} \le 1 + \sum_{n=2}^{N} \frac{1}{(n-1)^2} = 1 + \sum_{n=1}^{N} \frac{1}{n^2}$$

(consider upper sums with the partition $\{0,1,2,\ldots,N\}$). Since a_N is a bounded monotone sequence, it converges (by the monotone convergence theorem). Noting that a_N is also equal to $\int_{-N}^{0} \frac{1}{1+x^2}\,dx$, we are done.

Because the question said simply "explain" rather than "prove", you might have thought it was sufficient to say something like "it converges because each of the two components converges". □

Remark: If you want an extra challenge, try to prove that $\lim_{N\to\infty} a_N = 1 + \frac{\pi^2}{6}$.

(b) Explain why $\displaystyle\int_{-\infty}^{\infty} x\,dx$ does <u>not</u> exist, but $\displaystyle\lim_{N\to\infty}\int_{-N}^{N} x\,dx$ <u>does</u> exist.

Solution: The former is the sum of two divergent integrals, whereas the latter is 0 for all N.

(c) Prove that if $\displaystyle\int_{-\infty}^{\infty} f$ exists, then $\displaystyle\lim_{N\to\infty}\int_{-N}^{N} f$ exists and equals $\displaystyle\int_{-\infty}^{\infty} f$. Show, moreover, that $\displaystyle\lim_{N\to\infty}\int_{-N}^{N+1} f$ and $\displaystyle\lim_{N\to\infty}\int_{-N^2}^{N} f$ both exist and equal $\displaystyle\int_{-\infty}^{\infty} f$.

Can you state a reasonable generalization of these facts? (If you can't, you will have a miserable time trying to do these special cases!)

Solution: If $\displaystyle\int_{-\infty}^{\infty} f$ exists, then by definition each of $\displaystyle\int_{-\infty}^{0} f$ and $\displaystyle\int_{0}^{\infty} f$ must exist, which means $\displaystyle\lim_{N\to\infty}\int_{-N}^{0} f$ and $\displaystyle\lim_{N\to\infty}\int_{0}^{N} f$ both exist. Therefore,

$$\lim_{N\to\infty}\int_{-N}^{N} f = \lim_{N\to\infty}\left(\int_{-N}^{0} f + \int_0^N f\right) = \lim_{N\to\infty}\int_{-N}^{0} f + \lim_{N\to\infty}\int_0^N f = \int_{-\infty}^{0} f + \int_0^{\infty} f = \int_{-\infty}^{\infty} f$$

More generally, if $M(N)$ is an increasing function of N and $M(N) \to \infty$ as $N \to \infty$ then for any function $g(x)$ that converges to a limit at $x \to \infty$, we have $\lim_{N\to\infty} g(N) = \lim_{N\to\infty} g\big(M(N)\big)$. In particular, this is true for $g(x) = \int_0^x f$. $\square$

Remark: Intuitively, the issue here is that velocity along the real line is irrelevant provided we never slow down. (In fact, temporary slowing down—or even reversing direction—is OK provided $M(N)$ is eventually non-decreasing and unbounded.)

3. Prove that $|\sin x - \sin y| < |x - y|$ for all $x, y \in \mathbb{R}$ with $x \neq y$. *Hint:* The same statement, with $<$ replaced by $\leq$, is a straightforward consequence of a well-known theorem; simple supplementary considerations then allow $\leq$ to be improved to $<$.

   Solution: By the Mean Value Theorem, there exists ξ between x and y such that

   $$\left|\frac{\sin x - \sin y}{x - y}\right| = |\cos \xi| \ .$$

   and hence

   $$|\sin x - \sin y| = |\cos \xi|\,|x - y| \qquad (*)\,.$$

   Since $|\cos \xi| \leq 1$ for all $\xi \in \mathbb{R}$, we have

   $$|\sin x - \sin y| \leq |x - y| \ .$$

   Moreover, this inequality is strict unless ξ in equation (*) is such that $|\cos \xi| = 1$, *i.e.*, unless $\xi = k\pi$ for some $k \in \mathbb{Z}$.

   Suppose $x < y$. Choose any $w \in (x, y)$ such that the interval (x, w) does not contain $k\pi$ for any $k \in \mathbb{Z}$. Then there exist $\xi_1 \in (x, w)$ and $\xi_2 \in (w, y)$ such that

   $$\begin{aligned}\sin y - \sin x &= (\sin y - \sin w) + (\sin w - \sin x)\\ &= (\cos \xi_2)(y - w) + (\cos \xi_1)(w - x)\,.\end{aligned}$$

   Therefore, noting that $x < w < y$ implies $y - w > 0$ and $w - x > 0$, we have

   $$\begin{aligned}|\sin y - \sin x| &= |(\cos \xi_2)(y - w) + (\cos \xi_1)(w - x)|\\ &\leq |\cos \xi_2|\,(y - w) + |\cos \xi_1|\,(w - x) && \text{(triangle inequality)}\\ &\leq (y - w) + |\cos \xi_1|\,(w - x) && (|\cos \xi_2| \leq 1)\\ &< (y - w) + (w - x) && (|\cos \xi_1| < 1)\\ &= y - x\\ &= |y - x| \ .\end{aligned}$$

   The argument is similar if $y < x$. $\square$

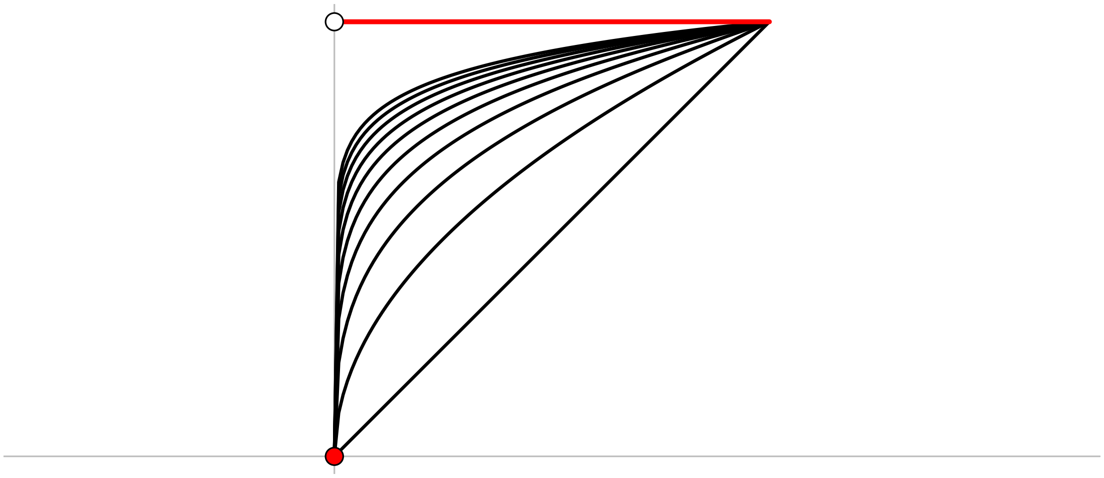
4. For each of the following sequences $\{f_n\}$, determine the pointwise limit of $\{f_n\}$ (if it exists) on the indicated interval, and establish whether $\{f_n\}$ converges uniformly to this function.

(i) $f_n(x) = \sqrt[n]{x}$, on $[0, 1]$;

Solution: The pointwise limit is

$$f(x) = \lim_{n\to\infty} \sqrt[n]{x} = \begin{cases} 0 & x = 0, \\ 1 & 0 < x \le 1, \end{cases}$$

which is not continuous. Therefore, the convergence is not uniform (if it were then $f(x)$ would be continuous). In the graph below, f_n is shown in black for $n = 1, \ldots, 10$ and f is shown in red.



(ii) $f_n(x) = \begin{cases} 0, & x \le n, \\ x - n & x \ge n, \end{cases}$ on $[a, b]$ and on $\mathbb{R}$;

Solution: The pointwise limit is $f(x) = 0$ for all $x \in \mathbb{R}$. On any finite interval $[a, b]$ this convergence is uniform. Note that for such a finite interval, we can find $N \in \mathbb{N}$ such that for all $n \ge N$, $f_n(x) = 0$ for all $x \in [a, b]$. However, the convergence is not uniform on $\mathbb{R}$ since each f_n is unbounded on $\mathbb{R}$, yet $f(x)$ is bounded. □

(iii) $f_n(x) = \dfrac{e^x}{x^n}$, on $(1, \infty)$.

Solution: Note that the numerator does not depend on n. Consequently, for any $x > 1$ we have the pointwise limit

$$f(x) = \lim_{n\to\infty} \frac{e^x}{x^n} = e^x \lim_{n\to\infty} \frac{1}{x^n} = 0\,.$$

On the other hand, for any given $n \in \mathbb{N}$ we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x}{x^n} = \infty\,,$$

so every f_n is unbounded. So, as in part (b), the convergence is not uniform. □

Version of December 8, 2016 @ 13:02.