Mathematics 3A03 Real Analysis I
2016 ASSIGNMENT 6

This assignment is **due in the appropriate locker** on **Wed 7 Dec 2016 at 2:25pm**.

1. Let f be integrable on $[a, b]$, let $c \in (a, b)$, and let

$$F(x) = \int_a^x f\,, \qquad a \le x \le b.$$

For each of the following statements, give either a proof or a counterexample.

(a) If f is differentiable at c, then F is differentiable at c.

(b) If f is differentiable at c, then F' is continuous at c.

(c) If f' is continuous at c, then F' is differentiable at c.

2. The ***improper integral*** $\int_{-\infty}^a f$ is defined in the obvious way, as

$$\lim_{N\to-\infty} \int_N^a f\,.$$

But another kind of improper integral $\int_{-\infty}^{\infty} f$ is defined in an unobvious way: it is

$$\int_{-\infty}^0 f + \int_0^{\infty} f\,,$$

provided these improper integrals both exist.

(a) Explain why $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ exists.

(b) Explain why $\int_{-\infty}^{\infty} x\,dx$ does <u>not</u> exist, but $\lim_{N\to\infty} \int_{-N}^{N} x\,dx$ <u>does</u> exist.

(c) Prove that if $\int_{-\infty}^{\infty} f$ exists, then $\lim_{N\to\infty} \int_{-N}^{N} f$ exists and equals $\int_{-\infty}^{\infty} f$. Show, moreover, that $\lim_{N\to\infty} \int_{-N}^{N+1} f$ and $\lim_{N\to\infty} \int_{-N^2}^{N} f$ both exist and equal $\int_{-\infty}^{\infty} f$.
Can you state a reasonable generalization of these facts? (If you can't, you will have a miserable time trying to do these special cases!)

3. Prove that $|\sin x - \sin y| < |x - y|$ for all $x, y \in \mathbb{R}$ with $x \neq y$. *Hint:* The same statement, with $<$ replaced by $\le$, is a straightforward consequence of a well-known theorem; simple supplementary considerations then allow $\le$ to be improved to $<$.

4. For each of the following sequences $\{f_n\}$, determine the pointwise limit of $\{f_n\}$ (if it exists) on the indicated interval, and establish whether $\{f_n\}$ converges uniformly to this function.

   (i) $f_n(x) = \sqrt[n]{x}$, on $[0, 1]$;

   (ii) $f_n(x) = \begin{cases} 0, & x \leq n, \\ x - n & x \geq n, \end{cases}$ on $[a, b]$ and on $\mathbb{R}$;

   (iii) $f_n(x) = \dfrac{e^x}{x^n}$, on $(1, \infty)$.

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