Mathematics 3A03 Real Analysis I

2016 ASSIGNMENT 5

SOLUTIONS

This assignment is **due in the appropriate locker** on **Fri 25 Nov 2016 at 4:25pm**.

1. (a) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Show that $f([a, b])$ is a closed interval. (Put another way: a continuous function maps compact intervals to compact intervals.) *Note:* Consider the single point $\{c\}$ to be the closed interal $[c, c]$.

   Solution: Since $[a, b]$ is compact, the extreme value theorem (EVT) implies that f attains maximum and minimum values on $[a, b]$, say M and m. Suppose $f(x_1) = m$ and $f(x_2) = M$, where $x_1, x_2 \in [a, b]$. If $x_1 \leq x_2$ then, since f is continuous on $[x_1, x_2]$, the intermediate value theorem (IVT) implies that for each $y \in [m, M]$, there exists $x \in [x_1, x_2]$ such that $f(x) = y$. But this means $f([a, b]) = [m, M]$, a closed interval. If $x_1 > x_2$ then the argument is the same based on $[x_2, x_1]$. □

   (b) Is it true that continuous functions map closed sets to closed sets? Is it true that continuous functions map open sets to open sets?

   Solution: Neither is true. Let $f(x) = 1/(1 + x^2)$. Then $f : \mathbb{R} \to (0, 1]$, *i.e.*, f maps a set that is both closed and open to a set that is neither closed nor open. Note also that a constant function maps any set to a closed set. □

2. Recall that a set A of real numbers is said to be ***dense*** if every open interval contains a point of A. For example, early in the course we showed in class that the set of rational numbers $\mathbb{Q}$ is dense.

   (a) Prove that if f is continuous and $f(x) = 0$ for all numbers x in a dense set A, then $f(x) = 0$ for all x.

   Solution: Pick any $x_0 \in \mathbb{R}$. f continuous at x_0 means that for any sequence in $\mathbb{R}$, if $x_n \to x_0$ then $f(x_n) \to f(x_0)$. Now, since A is dense, there is a sequence $\{a_n\}$ of points in A such that $a_n \to x_0$. We know $f(a_n) = 0$ for all n because $a_n \in A$ for all n. Therefore, $f(a_n) \to 0$. But since f is continuous at x_0, $f(a_n) \to f(x_0)$. Hence $f(x_0) = 0$. □

   (b) Prove that if f and g are continuous and $f(x) = g(x)$ for all x in a dense set A, then $f(x) = g(x)$ for all x.

   Solution: Apply part (a) to $h = f - g$. □

   (c) If we assume instead that $f(x) \geq g(x)$ for all x in the dense set A, show that $f(x) \geq g(x)$ for all x. Can $\geq$ be replaced by $>$ throughout?

   Solution: Let $h = f - g$. Since f and g are continuous, it follows that h is continuous. The condition $f(x) \geq g(x)$ is equivalent to $h(x) \geq 0$. Therefore, it is enough to prove that if h is continuous and $h(x) \geq 0$ for all $x \in A$ then $h(x) \geq 0$ for all $x \in \mathbb{R}$.

Pick any $x \in \mathbb{R}$ and choose a sequence $\{a_n\}$ in A such that $a_n \to x$. We know $h(a_n) \geq 0$ for all n, so $\lim_{n\to\infty} h(a_n) \geq 0$. But h is continuous at x, so $\lim_{n\to\infty} h(a_n) = h(x)$. Hence $h(x) \geq 0$. Since x was an arbitrary point in $\mathbb{R}$, $h(x) \geq 0$ for all $x \in \mathbb{R}$.

We cannot replace $\geq$ by $>$. For example, consider $h(x) = x^2$ and $A = \mathbb{R} \setminus \{0\}$. □

3. A function $f : [a,b] \to [a,b]$ is said to have a fixed point $c \in [a,b]$ if $f(c) = c$. Show that every continuous function f mapping $[a,b]$ into itself has at least one fixed point. *Hint:* Consider the function $g(x) = f(x) - x$.

   Solution: Note that since f is continous on $[a,b]$, so is g. Our goal is to show that g has a root in $[a,b]$, *i.e.,* $\exists c \in [a,b]$ such that $g(c) = 0$.

   Since the range of f is $[a,b]$, we know $a \leq f(x) \leq b$ for all $x \in [a,b]$. Therefore, $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$.

   If either $g(a) = 0$ or $g(b) = 0$ then we are done, so assume $g(a) > 0$ and $g(b) < 0$. Then the IVT implies g has a root between a and b. □

   Note: This is a special case of "Brouwer's fixed-point theorem".

4. (*Trapping principle.*) In class we considered the example of a function f defined in a neighbourhood I of 0 with the property that $|f(x)| \leq x^2$ for all $x \in I$. We showed that any such f is differentiable at 0 and $f'(0) = 0$. Suppose, more generally, that there is some function g defined on I such that $|f(x)| \leq g(x)$ for all $x \in I$.

   (a) Suppose $g(0) = 0$. What additional condition(s) on g are sufficient to guarantee that f is necessarily differentiable at 0? Propose and prove the most general theorem you can, *i.e.,* try to find the weakest sufficient additional condition(s) on g to ensure that $f'(0)$ exists.

   Solution: Since $|f(x)| \leq g(x)$ for all $x \in I$, we must have $g(x) \geq 0$ for all $x \in I$. Suppose

$$g \text{ is differentiable at } 0 \text{ and } g'(0) = 0, \qquad (\spadesuit)$$

   i.e.,

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = 0\,.$$

   Since differentiability implies continuity, we also know g is continuous at 0; given $g(0) = 0$ this means

$$\lim_{x\to 0} g(x) = g(0) = 0\,.$$

   In addition, $g(0) = 0$ implies $f(0) = 0$, so for all $x \neq 0$ we have

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{f(x)}{x}\right| \leq \left|\frac{g(x)}{x}\right| = \left|\frac{g(x) - g(0)}{x - 0}\right|\,.$$

From this, for all $x \neq 0$ we have

$$-\left|\frac{g(x)-g(0)}{x-0}\right| \leq \frac{f(x)-f(0)}{x-0} \leq \left|\frac{g(x)-g(0)}{x-0}\right|$$

Here, both the LHS and RHS $\to 0$ as $x \to 0$ (because the quantity inside the absolute value bars $\to 0$ as $x \to 0$). Consequently, the squeeze theorem implies that the quantity in the middle $\to 0$ as $x \to 0$, *i.e.,*

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0\,,$$

i.e., f is differentiable at 0 and $f'(0) = 0$.

(b) Are the sufficient condition(s) you found in part (a) also necessary?

Solution: We must have g differentiable at 0. If not, then $f = g$ satisfies $|f(x)| \leq g(x)$, yet f is not differentiable at 0. Given $g(0) = 0$, since $g(x) \geq 0$ for all x, 0 is a minimum point for g. Therefore, since g is differentiable at 0, we must have $g'(0) = 0$. Thus, given $g(0) = 0$, condition ($\spadesuit$) is both necessary and sufficient to ensure that f is differentiable at 0 and $f'(0) = 0$.

(c) What can be said if $g(0) \neq 0$? In particular, are the sufficient condition(s) you found still sufficient? If they were necessary with $g(0) = 0$, are they still necessary if $g(0) \neq 0$?

Solution: If $g(0) < 0$ then no function f can satisfy $|f(0)| \leq g(0)$, so we must have $g(0) \geq 0$. If $g(0) > 0$, then ($\spadesuit$) is not sufficient. For example, consider $g(x) \equiv 1$. This g is differentiable everywhere and $g'(0) = 0$, yet <u>any</u> function f bounded by ± 1 satisfies $|f(x)| \leq g(x)$.

The argument in part (b) showed that it is necessary that g is differentiable at 0, regardless of the value of $g(0)$. So suppose g is differentiable at 0 but $g(0) > 0$. In order to infer that any f satisfying $|f(x)| \leq g(x)$ for all x is differentiable at 0, is it necessary that $g'(0) = 0$, as in condition ($\spadesuit$)?

Since g is differentiable at 0, it is continuous at 0. Therefore, since $g(0) > 0$, the neighbourhood sign lemma implies that g is positive in some neighbourhood of 0. In fact, a slight modification of the proof of the neighbourhood sign lemma (take $\varepsilon = f(a)/2$ rather than $f(a)$) implies that $\exists m > 0$ and $\varepsilon > 0$ such that $g(x) > m$ for all $x \in (-\varepsilon, \varepsilon)$. Now consider any f satisfying $|f(x)| \leq g(x)$ for all x and, moreover, $|f(x)| \leq m$ for all $x \in (-\varepsilon, \varepsilon)$. An example of such of function is $f(x) = 0$ if $x \neq 0$ and $f(0) = m/2$, which is <u>not</u> differentiable at 0. So, in fact, no further condition on g can force all f's that satisfy $|f(x)| \leq g(x)$ for all x to be differentiable at 0. In particular, the sign of $g'(0)$ is irrelevant (so certainly not necessary). □

5. Use the Mean Value Theorem to prove the following.

(a) If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

Solution: Let a and b be any two points in the interval I, with $a < b$. By the MVT, $\exists\, \xi \in (a,b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi)\,. \qquad (\clubsuit)$$

But $f'(x) = 0$ for <u>all</u> $x \in I$, so in particular $f'(\xi) = 0$, and hence $f(b) = f(a)$. Since a and b were arbitrary points in the interval, f has the same value at every point in the interval. $\square$.

(b) If f and g are defined on the same interval and $f'(x) = g'(x)$ for all x in the interval, then there is some $c \in \mathbb{R}$ such that $f = g + c$.

Solution: Let $h = f - g$. Then h is differentiable and $h'(x) = 0$ for all x in the interval. Therefore, by part (a), h is constant, *i.e.,* $f = g + c$ for some $c \in \mathbb{R}$. $\square$

(c) If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.

Solution: Suppose $a, b \in I$ and $a < b$. By MVT $\exists\, \xi \in (a,b)$ such that ($\clubsuit$) holds. But $f'(x) > 0$ for all $x \in I$, so $f'(\xi) > 0$. Therefore $f(b) - f(a) > 0$, *i.e.,* $f(b) > f(a)$. Since this is true for any $a, b \in I$ with $a < b$, f is increasing on I. $\square$

6. (a) Prove that a bounded function $f : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there is a partition P of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon\,.$$

Solution: ($\Longrightarrow$) Suppose the bounded function f is integrable, *i.e.,* $\sup\{L(f,P) : P \text{ a partition of } [a,b]\} = \inf\{U(f,P) : P \text{ a partition of } [a,b]\} = \int_a^b f$. Given $\varepsilon > 0$, since $\int_a^b f$ is the least upper bound of the lower sums, there is a partition P_1 such that

$$\int_a^b f = \sup_{P'}\{L(f,P')\} < L(f,P_1) + \frac{\varepsilon}{2}\,,$$

i.e.,

$$-L(f,P_1) < -\int_a^b f + \frac{\varepsilon}{2}\,.$$

Similarly, there is a partition P_2 such that

$$U(f,P_2) < \inf_{P'}\{U(f,P')\} + \frac{\varepsilon}{2} = \int_a^b f + \frac{\varepsilon}{2}\,.$$

Let $P = P_1 \cup P_2$. Then

$$U(f,P) - L(f,P) \le U(f,P_2) - L(f,P_1) < \int_a^b f + \frac{\varepsilon}{2} - \int_a^b f + \frac{\varepsilon}{2} = \varepsilon\,.$$

($\Longleftarrow$) The function f is assumed bounded, so proving that f is integrable means establishing that $\sup_{P'}\{L(f,P')\} = \inf_{P'}\{U(f,P')\}$.
Given $\varepsilon > 0$, choose a partition P such that

$$U(f,P) - L(f,P) < \varepsilon\,.$$

Now, for any partition, and in particular for P, we have

$$L(f,P) \le \sup_{P'}\{L(f,P')\} \le \inf_{P'}\{U(f,P')\} \le U(f,P)\,,$$

and hence

$$0 \le \inf_{P'}\{U(f,P')\} - \sup_{P'}\{L(f,P')\} \le U(f,P) - L(f,P) < \varepsilon\,.$$

But by hypothesis, such a partition P can be found for <u>any</u> given $\varepsilon > 0$. Therefore $\inf_{P'}\{U(f,P')\} = \sup_{P'}\{L(f,P')\}$. □

(b) Suppose $b > 0$ and $f(x) = x$ for all $x \in \mathbb{R}$. Prove, using only the definition of the integral (or the result proved in part (a) of this question), that

$$\int_0^b f = \frac{b^2}{2}\,.$$

(This exercise should help you appreciate the Fundamental Theorem of Calculus.)
Solution: To apply the theorem proved in part (a), we need to show that for any given $\varepsilon > 0$ there is a partition P of $[0,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.
Let $P_n = \{t_0,\dots,t_n\}$ be a partition of $[0,b]$ into n subintervals of <u>equal</u> length. Thus, $t_i = ib/n$ for each $i = 0,1,\dots,n$. Then

$$\begin{aligned} L(f,P_n) &= \sum_{i=1}^{n} f(t_{i_1})(t_i - t_{i-1}) = \sum_{i=1}^{n} t_{i_1}(t_i - t_{i-1}) = \sum_{i=1}^{n} \frac{(i-1)b}{n}\cdot\frac{b}{n} \\ &= \frac{b^2}{n^2}\sum_{i=1}^{n}(i-1) = \frac{b^2}{n^2}\sum_{i=0}^{n-1} i = \frac{b^2}{n^2}\cdot\frac{(n-1)n}{2} = \frac{b^2}{2}\cdot\frac{(n-1)}{n}\,. \end{aligned}$$

Similarly,

$$U(f,P_n) = \frac{b^2}{2}\cdot\frac{(n+1)}{n}\,,$$

and hence

$$U(f,P_n) - L(f,P_n) = \frac{b^2}{n}\,.$$

Thus, given $\varepsilon > 0$, choose n large enough that $b^2/n < \varepsilon$. Then the partition $P = P_n$ satisfies $U(f,P) - L(f,P) < \varepsilon$. □

Note: If you're not yet convinced of the power of the Fundamental Theorem of Calculus, try computing $\int_0^b x^2\,dx$ directly from the definition of the integral.

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