Mathematics 3A03 Real Analysis I
2016 ASSIGNMENT 5

This assignment is **due in the appropriate locker** on **Fri 25 Nov 2016 at 4:25pm**.

1. (a) Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Show that $f([a, b])$ is a closed interval. (Put another way: a continuous function maps compact intervals to compact intervals.) *Note:* Consider the single point $\{c\}$ to be the closed interal $[c, c]$.
   (b) Is it true that continuous functions map closed sets to closed sets? Is it true that continuous functions map open sets to open sets?

2. Recall that a set A of real numbers is said to be ***dense*** if every open interval contains a point of A. For example, early in the course we showed in class that the set of rational numbers $\mathbb{Q}$ is dense.
   (a) Prove that if f is continuous and $f(x) = 0$ for all numbers x in a dense set A, then $f(x) = 0$ for all x.
   (b) Prove that if f and g are continuous and $f(x) = g(x)$ for all x in a dense set A, then $f(x) = g(x)$ for all x.
   (c) If we assume instead that $f(x) \geq g(x)$ for all x in the dense set A, show that $f(x) \geq g(x)$ for all x. Can $\geq$ be replaced by $>$ throughout?

3. A function $f : [a, b] \to [a, b]$ is said to have a fixed point $c \in [a, b]$ if $f(c) = c$. Show that every continuous function f mapping $[a, b]$ into itself has at least one fixed point. *Hint:* Consider the function $g(x) = f(x) - x$.

4. (*Trapping principle.*) In class we considered the example of a function f defined in a neighbourhood I of 0 with the property that $|f(x)| \leq x^2$ for all $x \in I$. We showed that any such f is differentiable at 0 and $f'(0) = 0$. Suppose, more generally, that there is some function g defined on I such that $|f(x)| \leq g(x)$ for all $x \in I$.
   (a) Suppose $g(0) = 0$. What additional condition(s) on g are sufficient to guarantee that f is necessarily differentiable at 0? Propose and prove the most general theorem you can, *i.e.,* try to find the weakest sufficient additional condition(s) on g to ensure that $f'(0)$ exists.
   (b) Are the sufficient condition(s) you found in part (a) also necessary?
   (c) What can be said if $g(0) \neq 0$? In particular, are the sufficient condition(s) you found still sufficient? If they were necessary with $g(0) = 0$, are they still necessary if $g(0) \neq 0$?

5. Use the Mean Value Theorem to prove the following.

   (a) If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

   (b) If f and g are defined on the same interval and $f'(x) = g'(x)$ for all x in the interval, then there is some $c \in \mathbb{R}$ such that $f = g + c$.

   (c) If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.

6. (a) Prove that a bounded function $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ if and only if for all $\varepsilon > 0$ there is a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon\,.$$

   (b) Suppose $b > 0$ and $f(x) = x$ for all $x \in \mathbb{R}$. Prove, using only the definition of the integral (or the result proved in part (a) of this question), that

$$\int_0^b f = \frac{b^2}{2}\,.$$

   (This exercise should help you appreciate the Fundamental Theorem of Calculus.)

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