Mathematics 3A01 Real Analysis I
2016 ASSIGNMENT 4
SOLUTIONS

This assignment is **due in the appropriate locker** on **Wed 9 Nov 2016 at 2:25pm**.

1. (a) Prove directly from the ε-δ definition that for any $a > 0$

$$\lim_{x\to a} \sqrt{1+x} = \sqrt{1+a}\,.$$

Solution: Since $a > 0$ we can restrict attention to $x > -1$ so there so $\sqrt{1+x}$ is well-defined. To be sure of this, we will restrict attention to $\varepsilon < 1$.
Given $0 < \varepsilon < 1$ we must find $\delta > 0$ such that if $|x-a| < \delta$ then $\left|\sqrt{1+x}-\sqrt{1+a}\right| < \varepsilon$. To this end, note that

$$\begin{aligned}
&\left|\sqrt{1+x}-\sqrt{1+a}\right| < \varepsilon \\
&\qquad\iff -\varepsilon < \sqrt{1+x}-\sqrt{1+a} < \varepsilon \\
&\qquad\iff \sqrt{1+a}-\varepsilon < \sqrt{1+x} < \varepsilon + \sqrt{1+a} \\
&\qquad\iff \left(\sqrt{1+a}-\varepsilon\right)^2 < 1+x < \left(\varepsilon+\sqrt{1+a}\right)^2 \\
&\qquad\iff \left(\sqrt{1+a}-\varepsilon\right)^2 - (1+a) < x-a < \left(\varepsilon+\sqrt{1+a}\right)^2 - (1+a)\,,
\end{aligned}$$

where we have used the fact that $f(x) = x^2$ is an increasing function on $[0,\infty)$ to infer that $0 \le \alpha < \beta < \gamma \implies 0 \le \alpha^2 < \beta^2 < \gamma^2$. Since the above steps are reversible, given $0 < \varepsilon < 1$ we can choose

$$\delta = \frac{1}{2}\min\left\{(1+a) - \left(\sqrt{1+a}-\varepsilon\right)^2,\ \left(\varepsilon+\sqrt{1+a}\right)^2 - (1+a)\right\}\,.$$

$\square$

(b) Use the theorem on limits of compositions of functions to calculate

$$\lim_{x\to 0} \sqrt{1+\sqrt{1+\sqrt{1+x}}}\,.$$

<u>Note</u>: *You must justify each step of your calculation.*

Solution: From part (a) we know that $f(x) = \sqrt{1+x}$ is continuous for all $x > 0$. In fact, the proof in part (a) works for any $x > -1$. So, in particular, $f(x) = \sqrt{1+x}$ is continuous at 0, so $\lim_{x\to 0}\sqrt{1+x} = f(0) = 1$. Now, since $f(x) = \sqrt{1+x}$ is continuous at 1, we have $\lim_{x\to 1}\sqrt{1+x} = \sqrt{2}$. Similarly, $\lim_{x\to\sqrt{2}}\sqrt{1+x} = \sqrt{1+\sqrt{2}}$. Putting this all together we have

$$\begin{aligned}
\sqrt{1+\sqrt{2}} &= \sqrt{1+\sqrt{1+1}} \\
&= \sqrt{1+\sqrt{1+\lim_{x\to 0}\sqrt{1+x}}} \\
&= \sqrt{1+\sqrt{\lim_{x\to 0}\left(1+\sqrt{1+x}\right)}} \\
&= \sqrt{1+\lim_{x\to 0}\sqrt{1+\sqrt{1+x}}} \\
&= \sqrt{\lim_{x\to 0}\left(1+\sqrt{1+\sqrt{1+x}}\right)} \\
&= \lim_{x\to 0}\sqrt{1+\sqrt{1+\sqrt{1+x}}}
\end{aligned}$$

2. In each part below, give an example of a function f that fails to be continuous at a point x_0, as described.

   (i) f is discontinuous merely because f is not defined at x_0;
   Solution: $f(x) = 0$ iff $x \neq 0$.
   Note: For simplicity, each example will be written with $x_0 = 0$. Defining $g(x) = f(x - x_0)$ will give an example about any $x_0 \in \mathbb{R}$.

   (ii) f is discontinuous because $\lim_{x\to x0} f(x)$ fails to exist;
   Solution:
   $$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

   (iii) f is discontinuous at x_0 even though neither defect (i) or (ii) occurs;
   Solution:
   $$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

   (iv) f is discontinuous at x_0 and discontinuous at infinitely many other points in a neighbourhood of x_0.
   Solution:
   $$f(x) = \begin{cases} 1 & x = 0 \text{ or } x = \frac{1}{n},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

In each part below, give an example of a function f that is continuous at x_0, but:

(v) is discontinuous at every point in a neighbourhood of x_0;

Solution:

$$f(x) = \begin{cases} x & x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

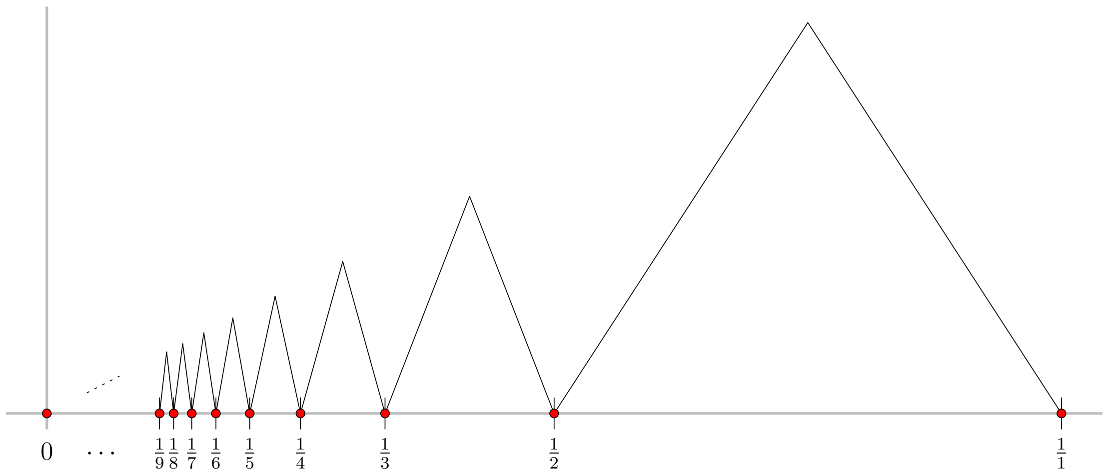
(vi) is discontinuous at countably infinitely many points in a neighbourhood of x_0;

Solution:

$$f(x) = \begin{cases} x & x = \frac{1}{n},\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

(vii) is continuous at countably infinitely many points in a neighbourhood $\mathcal{N}$ of x_0 and discontinous at all other points in $\mathcal{N}$.

Solution: We can construct such a function by combining aspects of each of the previous two examples. The following picture should help motivate the definition. The idea is that the only points of continuity are marked in red.



First define the sequence of midpoints between red points via

$$m_n = \frac{1}{2}\Big(\frac{1}{n+1} + \frac{1}{n}\Big)\,, \qquad n \in \mathbb{N}\,,$$

and the sequence of slopes via

$$s_n = \frac{m_n}{m_n - 1/(n+1)} = \frac{\frac{1}{2}\big(\frac{1}{n} + \frac{1}{n+1}\big)}{\frac{1}{2}\big(\frac{1}{n} - \frac{1}{n+1}\big)} = 2n+1\,, \qquad n \in \mathbb{N}\,.$$

Then define

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q}, \\ s_n\big(x - \frac{1}{n+1}\big) & x \in \mathbb{Q} \text{ and } \frac{1}{n+1} \le x < m_n,\ n \in \mathbb{N}, \\ m_n - s_n\big(x - m_n\big) & x \in \mathbb{Q} \text{ and } m_n \le x \le \frac{1}{n},\ n \in \mathbb{N}. \end{cases}$$

3. Suppose that f satisfies $f(x+y) = f(x) + f(y)$, and that f is continuous at 0. Prove that f is continuous at a for all $a \in \mathbb{R}$.

   Solution: Note that the question assumes implicitly that f is defined on all of $\mathbb{R}$.

   If we insert $x = y = 0$ in the given equation we have $f(0) = 2f(0)$, which implies $f(0) = 0$. Consequently, $0 = f\big(x + (-x)\big) = f(x) + f(-x)$, which implies $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

   Given that $f(0) = 0$, continuity at 0 means that $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $|x| < \delta$ then $|f(x)| < \varepsilon$.

   Continuity at an arbitrary given point $y \in \mathbb{R}$ means that $\forall \varepsilon > 0\ \exists \delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.

   Since f is continuous at 0, given $\varepsilon > 0$ choose $\delta > 0$ such that $|x| < \delta$ implies $|f(x)| < \varepsilon$. Given $y \in \mathbb{R}$, if $|x - y| < \delta$ then we have $|f(x-y)| < \varepsilon$. But $|f(x-y)| = |f(x) + f(-y)| = |f(x) - f(y)|$. Hence $|f(x) - f(y)| < \varepsilon$. Thus, f is continuous at y.

   Note that since δ does not depend on which y was chosen, f is actually uniformly continuous on $\mathbb{R}$. □

4. Prove that if continuity of g at L is not assumed, then it is not generally true that $\lim_{x \to x_0} g\big(f(x)\big) = g\big(\lim_{x \to x_0} f(x)\big)$.

   Solution: It should have been stated that $L = \lim_{x \to x_0} f(x)$, but hopefully this was clear from context.

   As an example, let $x_0 = 0$, $f(x) = x$, and

   $$g(x) = \begin{cases} 0 & x \neq 0, \\ 1 & x = 0. \end{cases}$$

   Then $L = \lim_{x \to 0} f(x) = 0$ so $g\big(\lim_{x \to 0} f(x)\big) = g(0) = 1$, but $g(f(x)) = g(x)$ for all x so $\lim_{x \to 0} g\big(f(x)\big) = \lim_{x \to 0} g(x) = 0$. □

5. (a) For which of the following values of α is the function $f(x) = x^\alpha$ uniformly continuous on $[0, \infty)$: $\alpha = \frac{1}{3}, \frac{1}{2}, 2, 3$?

   Solution: x^α is uniformly continuous on $[0, \infty)$ for $\alpha = \frac{1}{3}, \frac{1}{2}$ but not for $\alpha = 2, 3$. More generally, x^α is uniformly continuous on $[0, \infty)$ for $0 \leq \alpha \leq 1$ and not for $\alpha > 1$.

   First note that for any $\alpha \geq 0$ and any $\beta > 0$, x^α is uniformly continuous on the closed interval $[0, \beta]$ (because any continuous function on a compact set is uniformly continuous on that compact set). In particular, x^α is uniformly continuous on $[0, 1]$ for any $\alpha \geq 0$. This reduces the problem to determining for which $\alpha \geq 0$ the function x^α is uniformly continuous on $(1, \infty)$.

   Uniform continuity of x^α on $(1, \infty)$ means that $\forall \varepsilon > 0\ \exists \delta > 0$ such that, $\forall x, y \in (1, \infty)$, if $|x - y| < \delta$ then $|x^\alpha - y^\alpha| < \varepsilon$.

Consider $\alpha = 2$ and note that $|x^2 - y^2| = |(x-y)(x+y)| = |(x-y)|\,|(x+y)|$. No matter how small we make $|x-y|$, we can make $|x+y|$ as large as we like. More precisely, given any $\delta > 0$ and any $y \in (1,\infty)$, we can choose $x = y + \delta/2$. Then $|x-y| = \delta/2 < \delta$ but $|x^2 - y^2| = 2y + \delta/2 > y$. Since y is arbitrary in $(1,\infty)$, x^2 is not uniformly continuous on $(1,\infty)$.

For $\alpha = 3$, a similar argument based on $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ shows x^3 is not uniformly continuous on $(1,\infty)$. (*Use polynomial long division to compute* $(x^3-y^3)/(x-y)$ *if you don't remember the factorization.*)

Now consider $\alpha = 1$, which you weren't asked about but suggests the answer for $0 < \alpha < 1$. In this case uniform continuity on the entire real line is immediate: Given $\varepsilon > 0$ choose $\delta = \varepsilon$. Then $|x-y| < \delta \implies |x^1 - y^1| = |x-y| < \delta = \varepsilon$.

Now consider $\alpha = 1/2$. In this case, note that $|x-y| = \left|x^{1/2} - y^{1/2}\right|\left|x^{1/2} + y^{1/2}\right|$. Therefore, given $\varepsilon > 0$, choose $\delta = \varepsilon$ and note that if $x, y \in (1,\infty)$ and $|x-y| < \delta$ then

$$\left|x^{1/2} - y^{1/2}\right| = \frac{|x-y|}{|x^{1/2}+y^{1/2}|} < \frac{|x-y|}{2} = \frac{\delta}{2} < \delta = \varepsilon\,.$$

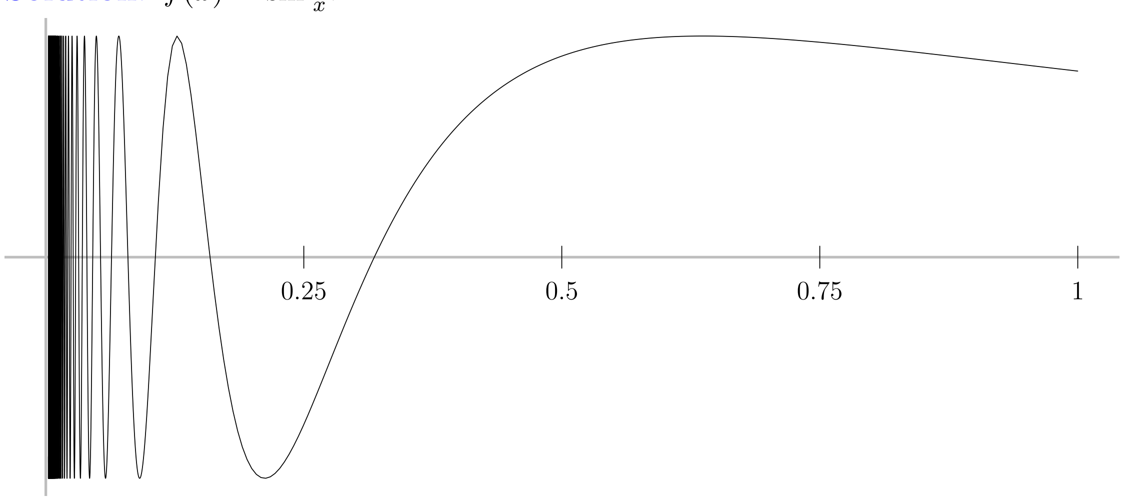
For $\alpha = 1/3$, note as above that $u^3 - v^3 = (u-v)(u^2+uv+v^2)$ and let $u = x^{1/3}$ and $v = y^{1/3}$ to obtain $x - y = (x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3})$. Then a similar argument shows $x^{1/3}$ is uniformly continuous on $(1,\infty)$. □

Although you were not asked about general α in the question, it is worth noting that a proof of the result for general $\alpha \in [0,\infty)$ will be much easier after we have proved the *Mean Value Theorem*.

Remark: An important consequence of this analysis of x^α is that a uniformly continuous function on an unbounded (and hence non-compact) set need not be bounded. In particular, a uniformly continuous function need not attain a maximum or minimum value (though the lack of a maximum or minimum can occur even for a uniformly continuous function on a bounded, non-compact set; what's an example?).

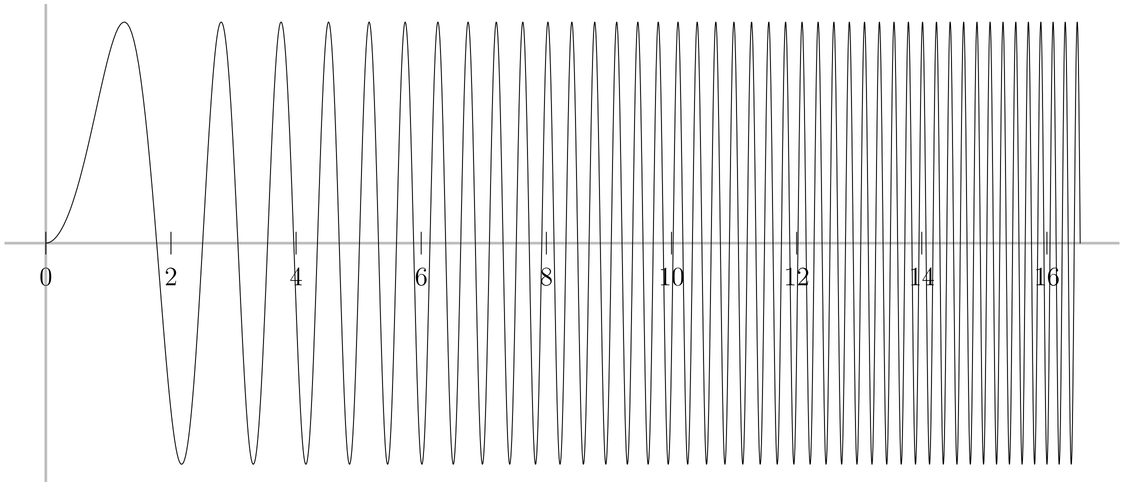
(b) Find a function f that is continuous and bounded on $(0,1]$, but not uniformly continuous on $(0,1]$.

Solution: $f(x) = \sin\frac{1}{x}$.



(c) Find a function f that is continuous and bounded on $[0, \infty)$ but which is not uniformly continuous on $[0, \infty)$.

Solution: $f(x) = \sin x^2$.



6. Prove that if f and g are each uniformly continuous on a set $E \subset \mathbb{R}$ then $f + g$ is also uniformly continuous on E.

Solution: Given $\varepsilon > 0$, choose $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon/2$ and $|g(x) - g(y)| < \varepsilon/2$. Then

$$\begin{aligned} |(f+g)(x) - (f+g)(y)| & \big|\big(f(x) + g(x)\big) - \big(f(y) + g(y)\big)\big| \\ &= \big|\big(f(x) - f(y)\big) + \big(g(x) - g(y)\big)\big| \\ &\le \big|\big(f(x) - f(y)\big)\big| + \big|\big(g(x) - g(y)\big)\big| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

□

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