Mathematics 3A01 Real Analysis I
2016 ASSIGNMENT 4

This assignment is **due in the appropriate locker** on **Wed 9 Nov 2016 at 2:25pm**.

1. (a) Prove directly from the ε-δ definition that for any $a > 0$

$$\lim_{x \to a} \sqrt{1+x} = \sqrt{1+a}\,.$$

   (b) Use the theorem on limits of compositions of functions to calculate

$$\lim_{x \to 0} \sqrt{1 + \sqrt{1 + \sqrt{1+x}}}\,.$$

   Note: You must justify each step of your calculation.

2. In each part below, give an example of a function f that fails to be continuous at a point x_0, as described.

   (i) f is discontinuous merely because f is not defined at x_0;
   (ii) f is discontinuous because $\lim_{x \to x0} f(x)$ fails to exist;
   (iii) f is discontinuous at x_0 even though neither defect (i) or (ii) occurs;
   (iv) f is discontinuous at x_0 and discontinuous at infinitely many other points in a neighbourhood of x_0.

   In each part below, give an example of a function f that is continuous at x_0, but:

   (v) is discontinuous at every point in a neighbourhood of x_0;
   (vi) is discontinuous at countably infinitely many points in a neighbourhood of x_0;
   (vii) is continuous at countably infinitely many points in a neighbourhood $\mathcal{N}$ of x_0 and discontinous at all other points in $\mathcal{N}$.

3. Suppose that f satisfies $f(x+y) = f(x) + f(y)$, and that f is continuous at 0. Prove that f is continuous at a for all $a \in \mathbb{R}$.

4. Prove that if continuity of g at L is not assumed, then it is not generally true that $\lim_{x \to x_0} g\big(f(x)\big) = g\big(\lim_{x \to x_0} f(x)\big)$.

5. (a) For which of the following values of α is the function $f(x) = x^\alpha$ uniformly continuous on $[0, \infty)$: $\alpha = \frac{1}{3}, \frac{1}{2}, 2, 3$?
   (b) Find a function f that is continuous and bounded on $(0, 1]$, but not uniformly continuous on $(0, 1]$.
   (c) Find a function f that is continuous and bounded on $[0, \infty)$ but which is not uniformly continuous on $[0, \infty)$.

6. Prove that if f and g are each uniformly continuous on a set $E \subset \mathbb{R}$ then $f + g$ is also uniformly continuous on E.

Version of November 1, 2016 @ 22:30.