Mathematics 3A01 Real Analysis I
2016 ASSIGNMENT 3
SOLUTIONS

This assignment is **due in the appropriate locker** on **Friday 21 Oct 2016 at 4:25pm**.

1. As you should always assume by default to be necessary, justify all your assertions when answering the following questions:

   (a) What can be said about the sequence $\{s_n\}$ if it converges and each s_n is an integer?

   Solution: Such a sequence must be "eventually constant" and converge to one of the terms in the sequence, *i.e.,* there must exist $N \in \mathbb{N}$ such that for all $n \geq N$, $s_n = s_N$. To see this, note that since $\{s_n\}$ converges, it is a Cauchy sequence. Therefore, there is some $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|s_n - s_m| < 1/2$. In particular, for all $n > N$, $|s_n - s_N| < 1/2$. But since s_n and s_N are both integers, this implies $s_n = s_N$ for all $n > N$. □

   (b) Find all convergent subsequences of the sequence $\{(-1)^n\}$. *Hint:* There are infinitely many, although there are only two limits that such subsequences can have.

   Solution: We know from part (a) that any convergent subsequence is eventually constant and converges to -1 or 1. Any finite sequence made up of -1's and 1's could precede the infinite sequence of -1's or 1's. □

   (c) Find all convergent subsequences of the sequence

   $$1,\ 2,\ 1,\ 2,\ 3,\ 1,\ 2,\ 3,\ 4,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$

   Hint: There are infinitely many limits that such subsequences can have.

   Solution: We know from part (a) that any convergent subsequence is eventually constant and converges to some $N \in \mathbb{N}$. Any finite sequence of natural numbers can precede the constant tail of the convergent subsequence. □

   (d) Consider the sequence

   $$\frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{6},\ \ldots$$

   For which numbers α is there a subsequence converging to α?

Solution: This sequence contains every rational number in the open interval $(0, 1)$, *i.e.,* every point in $(0, 1) \cap \mathbb{Q}$, and no other points. Moreover, each rational number in $(0, 1)$ occurs infinitely many times in the sequence, for instance:

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots$$

Consequently, any rational number in $(0, 1)$ can be the limit of a subsequence, in particular a constant subsequence equal to the limit. It is easy to see that the endpoints of the open interval $(0, 1)$ can also be the limits of subsequences, *e.g.,* $\{1/n\} \to 0$ and $\{(n-1)/n\} \to 1$. More generally, since every point in $(0, 1) \cap \mathbb{Q}$ occurs infinitely many times in the original sequence, we can construct *any* sequence of rational numbers between 0 and 1 as a subsequence. But $\mathbb{Q}$ is dense in $\mathbb{R}$, which implies any real number in $[0, 1]$ can be obtained as the limit of some subsequence of the original sequence. No point outside $[0, 1]$ can be the limit of a subsequence because there is a neighbourhood of any point outside $[0, 1]$ that contains no points of the given sequence. (Another way of saying all of this is that the closure of $(0, 1) \cap \mathbb{Q}$ is the closed interval $[0, 1]$.) □

2. (a) Prove that if a subsequence of a Cauchy sequence converges then so does the original Cauchy sequence.

Solution: Suppose $\{s_n\}$ is a Cauchy sequence with a convergent subsequence $\{s_{n_i}\}$, where $\{n_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$, and suppose the convergent subsequence converges to L. Given $\varepsilon > 0$ there exists $N_1 \in \mathbb{N}$ such that $|s_{n_i} - L| < \varepsilon/2$ for all $n_i \geq N_1$. In addition, since $\{s_n\}$ is a Cauchy sequence, there exists $N_2 \in \mathbb{N}$ such that $|s_n - s_m| < \varepsilon/2$ for all $n, m \geq N_2$. Let $N = \max\{N_1, N_2\}$ and let s_{n_i} be a term in the subsequence for which $n_i > N$. Then

$$|s_n - L| = |s_n - s_{n_i} + s_{n_i} - L| \leq |s_n - s_{n_i}| + |s_{n_i} - L| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

i.e., $\{s_n\}$ converges to L. □

(b) Prove that any subsequence of a convergent sequence converges.

Solution: A sequence of real numbers converges iff it is a Cauchy sequence. Hence consider a Cauchy sequence $\{s_n\}$. Given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|s_n - s_m| < \varepsilon$. Let $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$. Then, since the condition $|s_n - s_m| < \varepsilon$ holds for <u>all</u> pairs of terms s_n, s_m with $n, m \geq N$, it holds in particular for any such pairs of terms that happen to occur in the subsequence $\{s_{n_i}\}$. Hence the subsequence is also a Cauchy sequence. □

3. Determine which of the following sets are open, which are closed, and which are neither open nor closed.

   (a) $(-\infty, 0) \cup (0, \infty)$
   Solution: Open, not closed. It is a union of open intervals, hence open. The origin is an accumulation point that is not in the set, so it is not closed.

   (b) $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
   Solution: Not open, not closed. It contains no intervals so can't be open. It does not contain its accumulation point at 0, so it is not closed.

   (c) $\{0\} \cup \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
   Solution: Not open, closed. The missing accumulation point is now included.

   (d) $(0, 1) \cup (1, 2) \cup (2, 3) \cup (3, 4) \cup \cdots \cup (n, n + 1) \cup \cdots$
   Solution: Open, not closed. It is a union of open intervals, hence open. The set does not contain the accumulation points at the non-negative integers.

   (e) $(\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{2}) \cup (\frac{1}{8}, \frac{1}{4}) \cup (\frac{1}{16}, \frac{1}{8}) \cup \cdots$
   Solution: Open, not closed. It is a union of open intervals, hence open. The set does not contain the accumulation points at $1/n$ for each $n \in \mathbb{N}$ (nor does it contain the accumulation point at 0).

   (f) $\{x : |x - \pi| < 1\}$
   Solution: Open, not closed. This is the open interval $(\pi - 1, \pi + 1)$.

   (g) $\{x : x^2 < 2\}$
   Solution: Open, not closed. This is the open interval $(-\sqrt{2}, \sqrt{2})$.

   (h) $\mathbb{R} \setminus \mathbb{N}$
   Solution: Open, not closed. The complement $\mathbb{N}$ is closed, hence this set is open. Each point in $\mathbb{N}$ is an accumulation point of the set, but is not in the set, so the set is not closed.

   (i) $\mathbb{R} \setminus \mathbb{Q}$
   Solution: Not open, not closed. Any open interval containing an irrational number also contains a rational number, so the set is not open. Every point in $\mathbb{Q}$ is an accumulation point of $\mathbb{Q}^c$ so $\mathbb{Q}^c$ is not closed.

4. Prove or disprove: If $E \subseteq \mathbb{R}$ and E is both open and closed then $E = \mathbb{R}$ or $E = \varnothing$.

   Solution: The claim is true.

   As discussed in class, both $\mathbb{R}$ and $\varnothing$ are both open and closed. Suppose $E \neq \varnothing$ and E is both open and closed. We will show that $E = \mathbb{R}$.

   Since E is non-empty, it contains at least one point, say x. Since E is open, there is a neighbourhood of x that is contained in E. Note that any interval U containing x can be written as the union of two half-open intervals, $U = (x - \ell, x] \cup [x, x + r)$, where $\ell, r > 0$. Let

$$R = \sup \Big\{ r \in \mathbb{R} : [x, x + r) \subseteq E \Big\}, \qquad (*)$$

where we will use the notation $R = \infty$ if the least upper bound does not exist. If $R < \infty$ (*i.e.,* $R \in \mathbb{R}$) then—since E is closed—we must have $[x, x+R] = \overline{[x, x+R)} \subseteq E$. But then—since $x + R \in E$ and E is open—there is a neighbourhood of $x + R$ that is contained in E, contradicting R being the least upper bound in (*). Therefore, $R = \infty$. Now let

$$L = \inf\left\{\ell \in \mathbb{R} : (x - \ell, x] \subseteq E\right\}. \qquad (**)$$

Then, by a similar argument we must have $L = -\infty$. Thus, $(-\infty, \infty) \subseteq E$, *i.e.,* $E = \mathbb{R}$. □

5. Prove that a set E is

   (a) closed iff $\overline{E} = E$;

   Solution: For any set E, $\overline{E} = E \cup E'$, where E' is the set of accumulation points of E. By definition, a set is closed iff it contains all its accumulation points, *i.e.,* $E' \subseteq E$. Thus, we are asked to prove that

   $$E' \subseteq E \iff E \cup E' = E\,.$$

   ($\implies$) If $A \subseteq B$ then for any other set C, $C \cup A \subseteq C \cup B$. Therefore, $E' \subseteq E \implies E \cup E' \subseteq E \cup E = E$.

   ($\impliedby$) The meaning of $E \cup E' = E$ is that $E \cup E' \subseteq E$ and $E \cup E' \supseteq E$. But $E \cup E' \subseteq E$ implies that $E' \subseteq E$.

   □

   (b) open iff $E^\circ = E$.

   Solution: A set E is open iff for each point $x \in E$ there is a neighbourhood U of x such that $U \subseteq E$, *i.e.,* iff every point of E is an interior point of E, *i.e.,* iff the set of all interior points of E is entire set E, *i.e.,* iff $E^\circ = E$. □

6. Prove directly (*i.e.,* from the definition of the Bolzano-Weierstrass property) that

   (a) the interval $[0, \infty)$ does not have the Bolzano-Weierstrass property;

   Solution: We must show that there is some sequence of non-negative real numbers that either diverges or converges to a negative real number. For example, the sequence of natural numbers $\{n\}$ diverges to ∞. Note that it is not possible to find a sequence that converges to a point outside $[0, \infty)$ because $[0, \infty)$ is closed. □

(b) the union of two sets that with the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.

Solution:

Let $F = F_1 \cup F_2$, where F_1 and F_2 are sets with the Bolzano-Weierstrass property. Thus, for $i = 1$ or 2, any sequence in F_i contains a subsequence that converges to a point in F_i. Let $\{s_n\}$ be a sequence in F. The sequence $\{s_n\}$ must contain infinitely many terms in at least one of F_1 or F_2 (if not then there would be only finitely many points in the sequence), so assume without loss of generality that $\{s_n\}$ contains infinitely many points from F_1. Let $\{t_n\}$ be the subsequence of $\{s_n\}$ that contains only the points of $\{s_n\}$ that are in F_1. This is an infinite sequence in F_1 so—since F_1 has the Bolzano-Weierstrass property—$\{t_n\}$ contains a subsequence that converges to a point, say L, in F_1. But that subsequence of $\{t_n\}$ that converges to a point in F_1 is also a subsequence of the original sequence $\{s_n\}$ that converges to a point in F, as required. □

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