**Mathematics 3A01 Real Analysis I**
**2016 ASSIGNMENT 3**

This assignment is **due in the appropriate locker** on **Friday 21 Oct 2016 at 4:25pm**.

1. As you should always assume by default to be necessary, justify all your assertions when answering the following questions:
   (a) What can be said about the sequence $\{s_n\}$ if it converges and each $s_n$ is an integer?
   (b) Find all convergent subsequences of the sequence $\{(-1)^n\}$. *Hint:* There are infinitely many, although there are only two limits that such subsequences can have.
   (c) Find all convergent subsequences of the sequence
   $$1,\ 2,\ 1,\ 2,\ 3,\ 1,\ 2,\ 3,\ 4,\ 1,\ 2,\ 3,\ 4,\ 5,\ \ldots$$
   *Hint:* There are infinitely many limits that such subsequences can have.
   (d) Consider the sequence
   $$\frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{2}{4},\ \frac{3}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{6},\ \ldots$$
   For which numbers $\alpha$ is there a subsequence converging to $\alpha$?

2. (a) Prove that if a subsequence of a Cauchy sequence converges then so does the original Cauchy sequence.
   (b) Prove that any subsequence of a convergent sequence converges.

3. Determine which of the following sets are open, which are closed, and which are neither open nor closed.
   (a) $(-\infty,0)\cup(0,\infty)$
   (b) $\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\}$
   (c) $\{0\}\cup\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\ldots\}$
   (d) $(0,1)\cup(1,2)\cup(2,3)\cup(3,4)\cup\cdots\cup(n,n+1)\cup\cdots$
   (e) $(\frac{1}{2},1)\cup(\frac{1}{4},\frac{1}{2})\cup(\frac{1}{8},\frac{1}{4})\cup(\frac{1}{16},\frac{1}{8})\cup\cdots$
   (f) $\{x:|x-\pi|<1\}$
   (g) $\{x:x^2<2\}$
   (h) $\mathbb{R}\setminus\mathbb{N}$
   (i) $\mathbb{R}\setminus\mathbb{Q}$

4. Prove or disprove: If $E\subseteq\mathbb{R}$ and $E$ is both open and closed then $E=\mathbb{R}$ or $E=\varnothing$.

5. Prove that a set $E$ is
   (a) closed iff $\overline{E} = E$;
   (b) open iff $E^\circ = E$.
6. Prove directly (*i.e.*, from the definition of the Bolzano-Weierstrass property) that
   (a) the interval $[0, \infty)$ does not have the Bolzano-Weierstrass property;
   (b) the union of two sets that with the Bolzano-Weierstrass property must have the Bolzano-Weierstrass property.