Mathematics 3A01 Real Analysis I
2016 ASSIGNMENT 2
SOLUTIONS

This assignment is **due in the appropriate locker** on **Friday 30 Sep 2016 at 4:25pm**.

1. Suppose $m, n \in \mathbb{N}$. Prove that

   (a) if $m^2/n^2 < 2$ then $\dfrac{(m+2n)^2}{(m+n)^2} > 2$ and, furthermore,

$$\frac{(m+2n)^2}{(m+n)^2} - 2 \quad < \quad 2 - \frac{m^2}{n^2}\,; \tag{1}$$

Solution: First note that

$$\frac{(m+2n)^2}{(m+n)^2} > 2 \iff (m+2n)^2 > 2(m+n)^2 \tag{2a}$$

$$\iff m^2 + 4mn + 4n^2 > 2(m^2 + 2mn + n^2) \tag{2b}$$

$$\iff m^2 + \cancel{4mn} + 4n^2 > 2m^2 + \cancel{4mn} + 2n^2 \tag{2c}$$

$$\iff 2n^2 > m^2 \tag{2d}$$

$$\iff m^2/n^2 < 2 \tag{2e}$$

Thus, not only does $m^2/n^2 < 2$ imply $(m+2n)^2/(m+n)^2 > 2$, the two statements are actually equivalent.
Now note that inequality (1), which is the main thing that we are aiming to prove in this part, is equivalent to

$$\frac{(m+2n)^2 - 2(m+n)^2}{(m+n)^2} \quad < \quad \frac{2n^2 - m^2}{n^2}\,,$$

which, upon clearing fractions, is equivalent to

$$n^2[(m+2n)^2 - 2(m+n)^2] \quad < \quad (m+n)^2(2n^2 - m^2)\,.$$

But

$$(m+2n)^2 - 2(m+n)^2 = 2n^2 - m^2\,,$$

hence (1) is equivalent to

$$n^2(2n^2 - m^2) \quad < \quad (m+n)^2(2n^2 - m^2)\,,$$

i.e.,

$$0 \quad < \quad [(m+n)^2 - n^2](2n^2 - m^2)\,.$$

Since $(m+n)^2 > n^2$ for any $m, n \in \mathbb{N}$, this last inequality is true iff $2n^2 > m^2$, *i.e.*, iff $m^2/n^2 < 2$. □

(b) if $m^2/n^2 > 2$ then

$$\frac{(m+2n)^2}{(m+n)^2} - 2 \quad > \quad 2 - \frac{m^2}{n^2}\,;$$

Solution: The proof is the same as part (a) with all inequalities reversed. Note that although not requested specifically in this part, we first obtain $\frac{(m+2n)^2}{(m+n)^2} < 2$.

(c) if $m/n < \sqrt{2}$ then it is possible to write down a formula for another rational number m'/n' with

$$\frac{m}{n} < \frac{m'}{n'} < \sqrt{2}$$

(specifically, $m' = 3m + 4n$ and $n' = 2m + 3n$).

Solution: Given $m/n < \sqrt{2}$, let $\widetilde{m} = m + 2n$ and $\widetilde{n} = m + n$. Then part (a) yields $\widetilde{m}^2/\widetilde{n}^2 > 2$, *i.e.*, $\widetilde{m}/\widetilde{n} > \sqrt{2}$, and, furthermore,

$$0 \quad < \quad \frac{\widetilde{m}}{\widetilde{n}} - \sqrt{2} \quad < \quad \sqrt{2} - \frac{m}{n}\,. \tag{$\heartsuit$}$$

Note that this implies

$$0 \quad < \quad \left|\frac{\widetilde{m}}{\widetilde{n}} - \sqrt{2}\right| \quad < \quad \left|\frac{m}{n} - \sqrt{2}\right|\,,$$

i.e., $\widetilde{m}/\widetilde{n}$ is closer to $\sqrt{2}$ than m/n is. Now, since $\widetilde{m}/\widetilde{n} > \sqrt{2}$, letting $m' = \widetilde{m} + 2\widetilde{n}$ (*i.e.*, $m' = 3m+4n$) and $n' = \widetilde{m}+\widetilde{n}$ (*i.e.*, $n' = 2m+3n$), part (b) yields $m'/n' < \sqrt{2}$ and, furthermore,

$$\sqrt{2} - \frac{\widetilde{m}}{\widetilde{n}} \quad < \quad \frac{m'}{n'} - \sqrt{2} \quad < \quad 0\,.$$

Multiplying this inequality chain by -1 we have

$$0 \quad < \quad \sqrt{2} - \frac{m'}{n'} \quad < \quad \frac{\widetilde{m}}{\widetilde{n}} - \sqrt{2}\,. \tag{$\spadesuit$}$$

As above, note that this implies

$$0 \quad < \quad \left|\frac{m'}{n'} - \sqrt{2}\right| \quad < \quad \left|\frac{\widetilde{m}}{\widetilde{n}} - \sqrt{2}\right|\,,$$

i.e., m'/n' is closer to $\sqrt{2}$ than $\widetilde{m}/\widetilde{n}$ is. Putting ($\heartsuit$) and ($\spadesuit$) together we have, in particular,

$$0 \quad < \quad \sqrt{2} - \frac{m'}{n'} \quad < \quad \sqrt{2} - \frac{m}{n}\,.$$

Multiplying by -1 we have

$$\frac{m}{n} - \sqrt{2} \quad < \quad \frac{m'}{n'} - \sqrt{2} \quad < \quad 0\,.$$

Adding $\sqrt{2}$, we obtain

$$\frac{m}{n} \quad < \quad \frac{m'}{n'} \quad < \quad \sqrt{2}\,, \tag{3}$$

as required. □

2. Use the principle of mathematical induction to prove that for any $n \in \mathbb{N}$,

   (a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$;

   Solution: Let $P(n)$ denote the proposition that the stated formula is true. We must show that $P(1)$ is true and that if $P(n)$ is true then $P(n+1)$ is also true. Proposition $P(1)$ is simply $1 = 1$, which is true. Suppose $P(n)$ is true (this is called the ***induction hypothesis***). Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \Big(\sum_{k=1}^{n} k\Big) + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{(by the induction hypothesis)} \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+2)(n+1)}{2}\,,
\end{aligned}$$

   which confirms $P(n+1)$ given $P(n)$. Therefore, by the principle of mathematical induction, the proposition $P(n)$ is true for all $n \in \mathbb{N}$. □

   (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

   Solution: Let $P(n)$ denote the proposition that the stated formula is true. We must show that $P(1)$ is true and that if $P(n)$ is true then $P(n+1)$ is also true. Proposition $P(1)$ is simply $1 = 1$, which is true. Suppose $P(n)$ is true. Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= \Big(\sum_{k=1}^{n} k^2\Big) + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad \text{(by the induction hypothesis)} \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= (n+1)\frac{n(2n+1) + 6(n+1)}{6} \\
&= (n+1)\frac{2n^2 + 7n + 1}{6} \\
&= (n+1)\frac{2(n+1)^2 + 3(n+1) + 1}{6} \\
&= \frac{(n+1)\big((n+1)+1\big)\big(2(n+1)+1\big)}{6}\,,
\end{aligned}$$

   which confirms $P(n+1)$ given $P(n)$. Therefore, by the principle of mathematical induction, the proposition $P(n)$ is true for all $n \in \mathbb{N}$. □

Remark: If you are looking for a highly non-trivial Extra Challenge Problem, try to find a general formula (in terms of n and p) for

$$\sum_{k=1}^{n} k^p \,, \qquad p \in \mathbb{N}.$$

3. Use the formal definition of a limit of a sequence to prove that

(a) $\lim_{n\to\infty} \frac{2}{n^4} = 0$;

Solution: We need to show that given any $\varepsilon > 0$ there is a natural number N such that for all $n \geq N$, $2/n^4 < \varepsilon$. To figure out how to choose N given ε, suppose first that what we want to prove is true, *i.e.,* suppose $2/n^4 < \varepsilon$. Then $n^4 > 2/\varepsilon$, *i.e.,* $n > (2/\varepsilon)^{1/4}$. These steps are reversible, so we now have what we need to construct a proof:

Proof. Given $\varepsilon > 0$, let $N = \lceil (2/\varepsilon)^{1/4} \rceil + 1$ (where $\lceil x \rceil$ denotes the least integer greater or equal to than x). Then for any $n \geq N$, we have $n > (2/\varepsilon)^{1/4}$, which implies $n^4 > 2/\varepsilon$, *i.e.,* $2/n^4 < \varepsilon$. Hence $\lim_{n\to\infty} \frac{2}{n^4} = 0$. □

(b) $\lim_{n\to\infty} \frac{n^2 + 3n}{n^3 - 3} = 0$;

Solution: For any $n \in \mathbb{N}$, $n^2 + 3n < n^2 + 3n^2 = 4n^2$. Also, for any $n > 3$ we have $0 < n^3 - 3n^2 < n^3 - 3$, so $1/(n^3 - 3) < 1/(n^3 - 3n^2)$. Hence, for $n > 3$,

$$0 < \frac{n^2 + 3n}{n^3 - 3} < \frac{4n^2}{n^3 - 3} < \frac{4n^2}{n^3 - 3n^2} = \frac{4}{n - 3} \,,$$

so given $\varepsilon > 0$ it will be sufficient to choose N such that $4/(N - 3) < \varepsilon$, *i.e.,* $N > (4/\varepsilon) + 3$.

Proof. Given $\varepsilon > 0$, choose $N = \lceil 4/\varepsilon \rceil + 4$. Then $N > (4/\varepsilon) + 3$ and hence $4/(N - 3) < \varepsilon$. Moreover, since $N > 3$, given any $n \geq N$ we have

$$0 < \frac{n^2 + 3n}{n^3 - 3} < \frac{4n^2}{n^3 - 3} < \frac{4n^2}{n^3 - 3n^2} = \frac{4}{n - 3} < \frac{4}{N - 3} < \varepsilon.$$

Therefore, the stated limit is indeed 0. □

(c) $\lim_{n\to\infty} \left[\sqrt{n+1} - \left(\sqrt{n} + \sqrt{1} \right) \right] = -1$.

Solution: Intuitively, $\sqrt{n+1} \approx \sqrt{n}$ for large n, so it looks like the limit must indeed be -1. More formally, given $\varepsilon > 0$ we must show that for sufficiently large n,

$$\left| \left[\sqrt{n+1} - (\sqrt{n} + \sqrt{1}) \right] - (-1) \right| < \varepsilon \,.$$

But

$$\left| \left[\sqrt{n+1} - (\sqrt{n} + \sqrt{1}) \right] - (-1) \right| = \sqrt{n+1} - \sqrt{n} \,,$$

so our goal is to find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sqrt{n+1} - \sqrt{n} < \varepsilon\,.$$

To that end, note that for any $n \in \mathbb{N}$ we have

$$\begin{aligned} 0 < \sqrt{n+1} - \sqrt{n} &= \left[\sqrt{n+1} - \sqrt{n}\right] \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &< \frac{1}{\sqrt{n} + \sqrt{n}} \\ &= \frac{1}{2\sqrt{n}}\,. \end{aligned}$$

Noting now that $1/(2\sqrt{N}) < \varepsilon$ iff $N > 1/(4\varepsilon^2)$, we're in business:

Proof. Given $\varepsilon > 0$, let $N = \lceil 1/(4\varepsilon^2) \rceil + 1$. Then, for any $n \geq N$, $1/(2\sqrt{n}) \leq 1/(2\sqrt{N}) < \varepsilon$. Moreover, from the analysis above, we have that for any $n \in \mathbb{N}$, $\sqrt{n+1} - \sqrt{n} < 1/(2\sqrt{n})$. Consequently, for any $n \geq N$, we have

$$\left|\left[\sqrt{n+1} - (\sqrt{n} + \sqrt{1})\right] - (-1)\right| = \sqrt{n+1} - \sqrt{n} < \varepsilon,$$

as required. □

4. Use the formal definition to prove that the following sequences $\{s_n\}$ <u>diverge</u> as $n \to \infty$.

   (a) $s_n = r^n$ (for any $r > 1$) ;

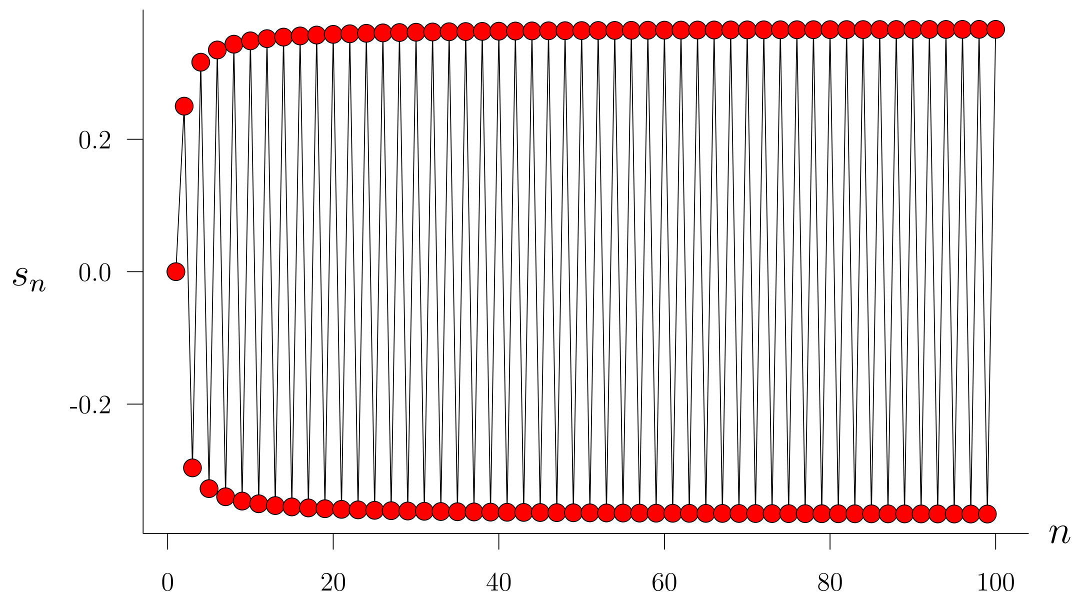
   Solution: If $M > 0$ then $r^n > M \iff n \log r > \log M \iff n > \log r / \log M$. So, given $M > 0$, if we choose $N > \log r / \log M$ then for any $n \geq N$ we will have $r^n > M$. Here, we are using the fact that $\log r$ is well-defined and positive for any real number $r > 1$, which is not something we've proved. Can we do without the logarithm function?

   Let $\delta = r - 1$ and note that $\delta > 0$ since $r > 1$. Thus $r^n = (1+\delta)^n$. To find a suitable $N \in \mathbb{N}$, note that $(1+\delta)^n > 1 + n\delta$ for all n (this is easy to prove by induction). Consequently we can make $(1+\delta)^n$ as large as we like by making $1+\delta n$ sufficiently large. Now we can construct a proof without using the a function (log) that we haven't defined:

   Proof. Given any $M > 0$, let $N = \lceil M/\delta \rceil$, where $\delta = r - 1 > 0$. Then $N \geq M/\delta > (M-1)/\delta$, and hence $\delta N > M - 1$, *i.e.*, $1 + \delta N > M$. Consequently, for any $n \geq N$, it follows that $r^n = (1+\delta)^n > 1 + \delta n \geq 1 + \delta N > M$. Thus, if $r > 1$ then the sequence $\{r^n\}$ diverges to ∞. □

(b) $s_n = \left(\frac{1}{n} - 1\right)^n$.

Oops!!! This problem is too hard at this stage. My apologies! Hopefully you found it interesting to think about. We will return to this problem later in the course. For now, here's a plot:



5. Suppose $L \in \mathbb{R}$ and

$$\lim_{n\to\infty} s_n = L.$$

Use the formal definition of a limit of a sequence to prove that

$$\lim_{n\to\infty} s_n^2 = L^2.$$

Solution: One approach would be to replicate the proof given in class concerning the limit of a product of two convergent sequences. The present problem is the special case of that theorem when both sequences are the same. But let's instead reason this through from scratch without appealing to the proof of the more general result. Given $\varepsilon > 0$, we must show that for sufficiently large n, $|s_n^2 - L^2| < \varepsilon$. Let's manipulate $|s_n^2 - L^2|$ in ways that might allow us to use the fact that we can make $|s_n - L|$ as small as we like:

$$\left|s_n^2 - L^2\right| = |(s_n - L)(s_n + L)| = |s_n - L|\,|s_n + L| \ .$$

We can make $|s_n - L|$ as small as we like for sufficiently large n, so, in particular, we can make it less than $|L|$, in which case $|s_n + L| = |s_n - L + 2L| \leq |s_n - L| + 2\,|L| < 3\,|L|$. Now, for sufficiently large n, we can ensure that $|s_n - L| < \varepsilon/(3\,|L|)$. This is everything we need for our proof:

Proof. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|s_n - L| < \min\left\{|L|, \frac{\varepsilon}{3|L|}\right\}.$$

Then, for all $n \geq N$, we have

$$\begin{aligned}
\left|s_n^2 - L^2\right| &= |(s_n - L)(s_n + L)| = |s_n - L|\,|s_n + L| \\
&= |s_n - L|\,|s_n - L + 2L| \leq |s_n - L|\,\big(|s_n - L| + 2|L|\big) \\
&< |s_n - L|\,\big(|L| + 2|L|\big) \\
&= |s_n - L|\,\big(3|L|\big) \\
&< \frac{\varepsilon}{3|L|}\big(3|L|\big) = \varepsilon,
\end{aligned}$$

as required. □

6. Problem 1 showed that if m/n is a rational approximation to $\sqrt{2}$ then $(m+2n)/(m+n)$ is a better approximation. This implies that starting from any rational number q, we can construct a sequence of rational numbers that gets closer and closer to $\sqrt{2}$. In particular, if we start with $m = n = 1$ then we obtain

$$1, \ \frac{3}{2}, \ \frac{7}{5}, \ \frac{17}{12}, \ \ldots$$

(a) Prove that this sequence is given recursively by

$$q_1 = 1, \qquad q_{n+1} = 1 + \frac{1}{1 + q_n}.$$

Solution: The sequence of rational numbers that arise in problem 1 is

$$\underbrace{\frac{m}{n}}_{q_1} \quad \to \quad \underbrace{\frac{\widetilde{m}}{\widetilde{n}} = \frac{m + 2n}{m + n}}_{q_2} \quad \to \quad \underbrace{\frac{m'}{n'} = \frac{\widetilde{m} + 2\widetilde{n}}{\widetilde{m} + \widetilde{n}}}_{q_3} \quad \to \quad \cdots. \tag{4}$$

Observe that

$$\frac{m + 2n}{m + n} = \frac{(m/n) + 2}{(m/n) + 1} = \frac{[(m/n) + 1] + 1}{(m/n) + 1} = 1 + \frac{1}{1 + (m/n)}.$$

Thus, if we define

$$f(x) = 1 + \frac{1}{1 + x},$$

then for any $q \in \mathbb{Q}$ the sequence is given recursively by $q \to f(q) \to f(f(q)) \to \cdots$. In particular, if the first term of the sequence is 1 then we have

$$q_1 = 1, \qquad q_{n+1} = 1 + \frac{1}{1 + q_n}, \quad n \in \mathbb{N},$$

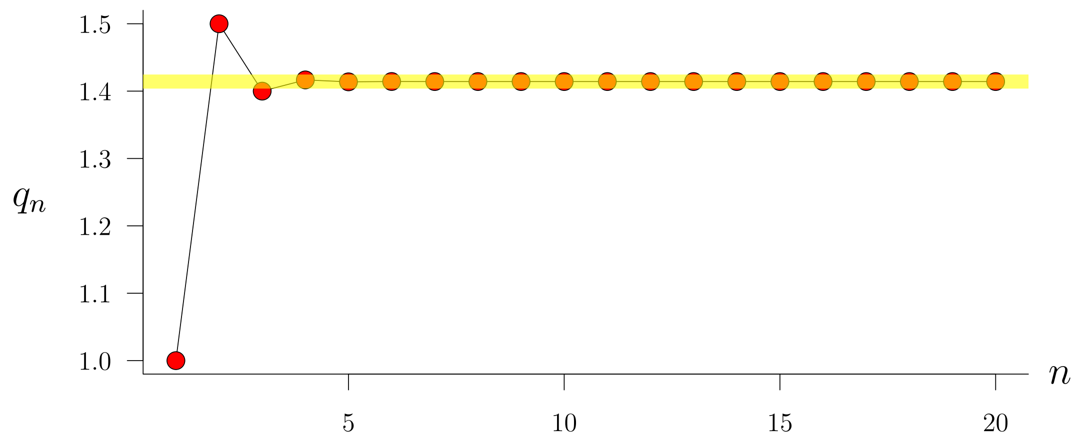
as required. □

(b) Prove that

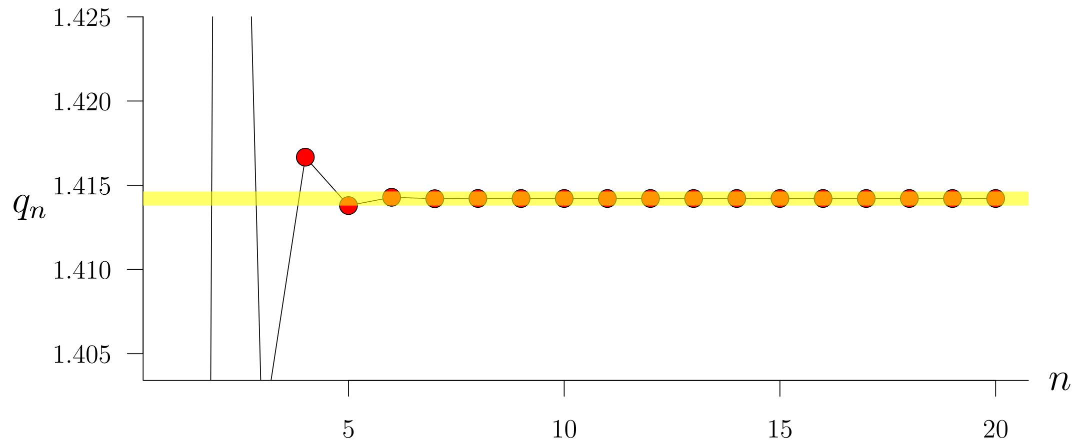
$$\lim_{n\to\infty} q_n = \sqrt{2}\,. \qquad (*)$$

Hint: Separately consider the subsequences $\{q_{2n}\}$ and $\{q_{2n+1}\}$ and show that they both converge to the same limit.

Solution: It is often helpful to make a plot of a sequence to see what's going on and potentially help us discover a way to proceed with a proof. Let's plot the sequence and highlight the claimed limit in yellow:

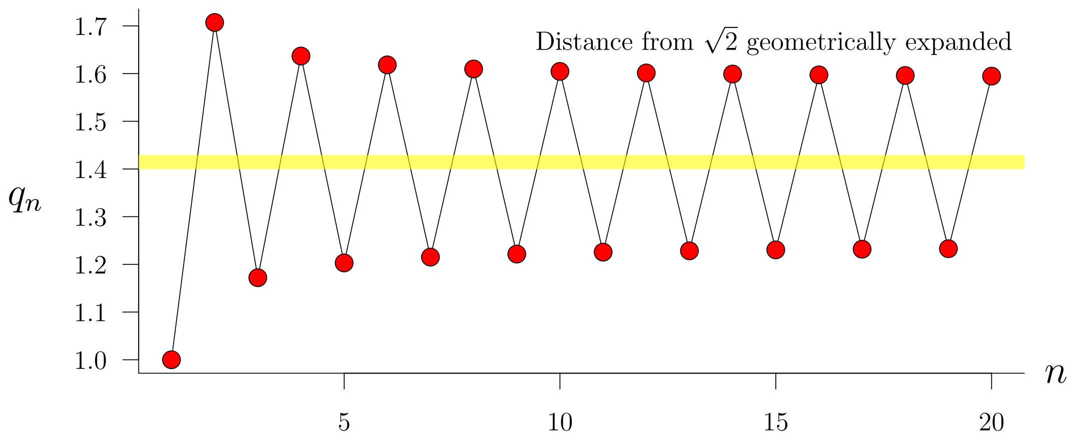


It certainly does appear to converge to $\sqrt{2}$, but the convergence is so rapid that it is hard to see how it gets there. Let's replot with vertical axis limits $\sqrt{2} \pm 0.01$.



This didn't help much, because the convergence is so fast. We'll have to stretch the distance between the sequence elements and the limit. So let's plot

$$\sqrt{2} + \mathrm{sign}(q_n - \sqrt{2}) \left| q_n - \sqrt{2} \right|^{1/n} .$$



Now we can see clearly, as we know analytically from problem 1, that the sequence alternates between terms that are below and above $\sqrt{2}$. (Moreover, as a matter of interest, the fact that the distance from $\sqrt{2}$ appears to be constant in the graph above suggests strongly that the rate of convergence to the limit is geometric; consequently, extremely accurate approximations of $\sqrt{2}$ can be obtained by iterating only a few terms of this sequence.)

The graphs produced by our numerical exploration support the proposed approach suggested in the hint. Namely, we should separately consider subsequences made up of even and odd numbered terms.

Proof. We will make use of the notation established in part (a) (*e.g.,* (4)) and the analysis conducted in Problem 1. Since $q_1 = 1$, we certainly have $q_1 < \sqrt{2}$, so inequality (2) implies that $q_2 > \sqrt{2}$; in addition, inequality (3) implies

$$q_1 < q_3 < \sqrt{2}\,. \tag{5}$$

Moreoever, by an argument analogous to that yielding inequality (3), but starting instead from $m/n > \sqrt{2}$, we obtain

$$\sqrt{2} < q_4 < q_2\,. \tag{6}$$

Thus, it follows more generally that

$$q_1 < q_3 < q_5 < \cdots < \sqrt{2} < \cdots < q_6 < q_4 < q_2\,, \tag{7}$$

which can be expressed as

$$q_{2n+1} \;<\; q_{2n+3} \;<\; \sqrt{2} \;<\; q_{2n+2} \;<\; q_{2n}\,, \qquad n \in \mathbb{N}, \tag{8}$$

i.e., the odd-numbered subsequence $\{q_{2n-1}\}$ is increasing and bounded above by $\sqrt{2}$ and the even-numbered subsequence $\{q_{2n}\}$ is decreasing and bounded below by $\sqrt{2}$. Therefore, by the monotone convergence theorem, both these subsequences converge (to their least upper and greatest lower bound, respectively). Let

$$L = \lim_{n\to\infty} q_{2n-1}\,, \qquad R = \lim_{n\to\infty} q_{2n}\,. \tag{9}$$

Inequalities (7), together with the theorem that limits of sequences retain bounds, imply that

$$L \leq \sqrt{2} \leq R\,. \tag{10}$$

What remains to show is that $L = R$. This is equivalent to showing $R - L = 0$, so consider the differences between corresponding terms in the even-numbered and odd-numbered sequences,

$$q_{2n+2} - q_{2n+1} = \left(1 + \frac{1}{1+q_{2n+1}}\right) - \left(1 + \frac{1}{1+q_{2n}}\right) = \frac{q_{2n+2} - q_{2n+1}}{(1+q_{2n+1})(1+q_{2n})}\,. \tag{11}$$

Since the sequences $\{q_{2n}\}$ and $\{q_{2n+1}\}$ converge (and neither L nor R is -1) we can use the theorem on the algebra of limits to infer that the limit of the LHS of equation (11) is

$$\lim_{n\to\infty}(q_{2n+2} - q_{2n+1}) = \lim_{n\to\infty} q_{2n+2} - \lim_{n\to\infty} q_{2n+1} = R - L\,, \tag{12}$$

and the limit of the RHS of equation (11) is

$$\lim_{n\to\infty} \frac{q_{2n+2} - q_{2n+1}}{(1+q_{2n+1})(1+q_{2n})} = \frac{\lim_{n\to\infty} q_{2n+2} - \lim_{n\to\infty} q_{2n+1}}{(1+\lim_{n\to\infty} q_{2n+1})(1+\lim_{n\to\infty} q_{2n})} = \frac{R-L}{(1+L)(1+R)}\,. \tag{13}$$

Thus

$$R - L = \frac{R-L}{(1+L)(1+R)}\,, \tag{14}$$

which is possible iff $R = L$. Given inequality (10), we finally have $L = R = \sqrt{2}$. $\square$

It is worth noting that (*) implies that

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\cdots}}\,,$$

which is called the *continued fraction expansion* of $\sqrt{2}$.

Remark: As an Extra Challenge Problem you might like to use the ideas in this problem to prove that the square root of any natural number can be well-approximated by a continued fraction. *Big hint:* Prove that for any $m, n \in \mathbb{N}$,

$$\sqrt{m^2+n} \quad = \quad m + \frac{n}{2m + \frac{n}{2m+\cdots}}\,.$$

Now, given a natural number N, what choice for m and n in the continued fraction yields the fastest convergence to $\sqrt{N}$, *i.e.*, if you want to approximate $\sqrt{N}$ to a given number of decimal places using the fewest possible iterations of the sequence, how should you choose m and n?