Mathematics 3A01 Real Analysis I
2016 ASSIGNMENT 2

This assignment is **due in the appropriate locker** on **Friday 30 Sep 2016 at 4:25pm**.

1. Suppose $m, n \in \mathbb{N}$. Prove that

   (a) if $m^2/n^2 < 2$ then $\dfrac{(m+2n)^2}{(m+n)^2} > 2$ and, furthermore,
   $$\frac{(m+2n)^2}{(m+n)^2} - 2 \quad < \quad 2 - \frac{m^2}{n^2}\,;$$

   (b) if $m^2/n^2 > 2$ then
   $$\frac{(m+2n)^2}{(m+n)^2} - 2 \quad > \quad 2 - \frac{m^2}{n^2}\,;$$

   (c) if $m/n < \sqrt{2}$ then it is possible to write down a formula for another rational number m'/n' with
   $$\frac{m}{n} < \frac{m'}{n'} < \sqrt{2}$$
   (specifically, $m' = 3m + 4n$ and $n' = 2m + 3n$).

2. Use the principle of mathematical induction to prove that for any $n \in \mathbb{N}$,

   (a) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$;

   (b) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

3. Use the formal definition of a limit of a sequence to prove that

   (a) $\displaystyle\lim_{n\to\infty} \frac{2}{n^4} = 0$;

   (b) $\displaystyle\lim_{n\to\infty} \frac{n^2+3n}{n^3-3} = 0$;

   (c) $\displaystyle\lim_{n\to\infty} \left[\sqrt{n+1} - \left(\sqrt{n} + \sqrt{1}\right)\right] = -1$.

4. Use the formal definition to prove that the following sequences $\{s_n\}$ *diverge* as $n \to \infty$.

   (a) $s_n = r^n$ (for any $r > 1$) ;

   (b) $s_n = \left(\dfrac{1}{n} - 1\right)^n$.

5. Suppose $L \in \mathbb{R}$ and

$$\lim_{n\to\infty} s_n = L.$$

Use the formal definition of a limit of a sequence to prove that

$$\lim_{n\to\infty} s_n^2 = L^2.$$

6. Problem 1 showed that if m/n is a rational approximation to $\sqrt{2}$ then $(m+2n)/(m+n)$ is a better approximation. This implies that starting from any rational number q, we can construct a sequence of rational numbers that gets closer and closer to $\sqrt{2}$. In particular, if we start with $m = n = 1$ then we obtain

$$1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \ldots$$

(a) Prove that this sequence is given recursively by

$$q_1 = 1, \qquad q_{n+1} = 1 + \frac{1}{1+q_n}\,.$$

(b) Prove that

$$\lim_{n\to\infty} q_n = \sqrt{2}\,. \tag{*}$$

Hint: Separately consider the subsequences $\{q_{2n}\}$ and $\{q_{2n+1}\}$ and show that they both converge to the same limit.

It is worth noting that (*) implies that

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2+\cdots}}\,,$$

which is called the *continued fraction expansion* of $\sqrt{2}$.