Mathematics 3A01 Real Analysis I
2016 ASSIGNMENT 1
SOLUTIONS

This assignment is **due in the appropriate locker** on **Friday 16 Sep 2016 at 4:25pm**.

1. Prove that $\sqrt{3}$ is irrational.

   Solution: As a first step, we'll prove that an integer m is a multiple of 3 *if and only if* m^2 is a multiple of 3. To establish this, we need to prove both the "if" and "only if" directions of this statement.

   "Only if" direction: If m is a multiple of 3 then there is another integer k such that $m = 3k$, which implies that $m^2 = 9k^2 = 3(3k^2)$, *i.e.*, m^2 is also a multiple of 3.

   "If" direction: If m is not a multiple of a 3 then either $m = 3k+1$ or $m = 3k+2$ for some integer k. In either of these cases $m^2 = 3\ell + 1$ for some integer ℓ and hence is not a multiple of 3. To see this, note that $(3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ and $(3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k) + 4 = 3(3k^2 + 4k + 1) + 1$.

   Thus m is a multiple of 3 if and only if m^2 is a multiple of 3.

   Now suppose, in order to derive a contradiction, that $\sqrt{3} \in \mathbb{Q}$. Then there exist two positive integers m and n with $\gcd(m, n) = 1$ such that $m/n = \sqrt{3}$.

   $$\therefore \qquad \left(\frac{m}{n}\right)^2 = \left(\sqrt{3}\right)^2 \quad \Longrightarrow \quad \frac{m^2}{n^2} = 3 \quad \Longrightarrow \quad m^2 = 3n^2.$$

   Thus, m^2 is a multiple of 3, and hence—from our analysis above—it follows that m is a multiple of 3. Therefore, $m = 3k$ for some $k \in \mathbb{N}$, which implies $m^2 = 9k^2 = 3n^2$, and hence $3k^2 = n^2$. Thus, n^2 is a multiple of 3, which implies—again from our analysis above—that n in a multiple of 3.

   We have now established that both m and n contain a factor of 3, which contradicts $\gcd(m, n) = 1$. Our initial assumption that $\sqrt{3} \in \mathbb{Q}$ must therefore be false, and we can conclude that $\sqrt{3} \notin \mathbb{Q}$. □

2. The field of integers modulo 2 ($\mathbb{Z}_2$) can be defined by interpretting "number" to mean either 0 or 1, and $+$ and $\cdot$ to be the operations specified by the following two tables.

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

   Prove that all the field axioms hold for $\mathbb{Z}_2$, even though $1 + 1 = 0$.

   Solution: The field axioms are given on slide 11 of Lecture 2. The addition and multiplcation tables above give the results of adding or multiplying all possible pairs of elements of $\mathbb{Z}_2$. Consequently, to verify the axioms that involve only pairs of elements of the field, we can simply check the table entries. This works for A1 (closed, commutative), A3 (additive identity), A4 (additive inverses), M1 (closed, commutative), M3 (multiplicative identity), M4 (multiplicative inverses). What remain to verify are the associative laws (A2 and M2) and the distributive law (AM1).

There are eight cases for A2, since each of x, y, z can be 0 or 1. But we can reduce the number of cases that must be checked by exploiting the other axioms that have already been verified. In particular, because A3 is true, it follows that $x + (y + z) = (x + y) + z$ if x, y or z is 0, so only the case $x = y = z = 1$ must be checked. Similarly for M2. Finally, AM1 is true for $x = 0$, since $0 \cdot y = 0$ for all $y \in \mathbb{Z}_2$, and it is true for $x = 1$, since $1 \cdot y = y$ for all y. $\square$

3. For each of the following sets, find the greatest lower bound (inf), least upper bound (sup), minimum (min) and maximum (max), if they exist, or indicate non-existence ($\nexists$). Justify your assertions.

   (a) $(-2, -1)$.

   (b) $\{\frac{1}{n} : n \in \mathbb{N}\}$.

   (c) $\{\frac{1}{n} : n \in \mathbb{Z} \text{ and } n \neq 0\}$.

   (d) $\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\}$.

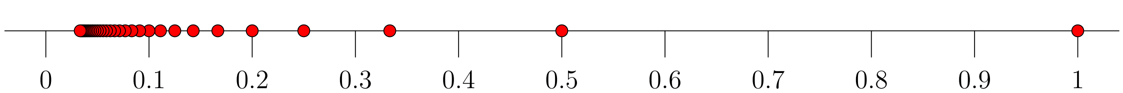
   Solution: The answers to the questions are most easily summarized in a table:

	Set	inf	sup	min	max
(a)	$(-2, -1)$	-2	-1	$\nexists$	$\nexists$
(b)	$\{\frac{1}{n} : n \in \mathbb{N}\}$	0	1	$\nexists$	1
(c)	$\{\frac{1}{n} : n \in \mathbb{Z} \text{ and } n \neq 0\}$	-1	1	-1	1
(d)	$\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\}$	-1	$3/2$	$\nexists$	$3/2$

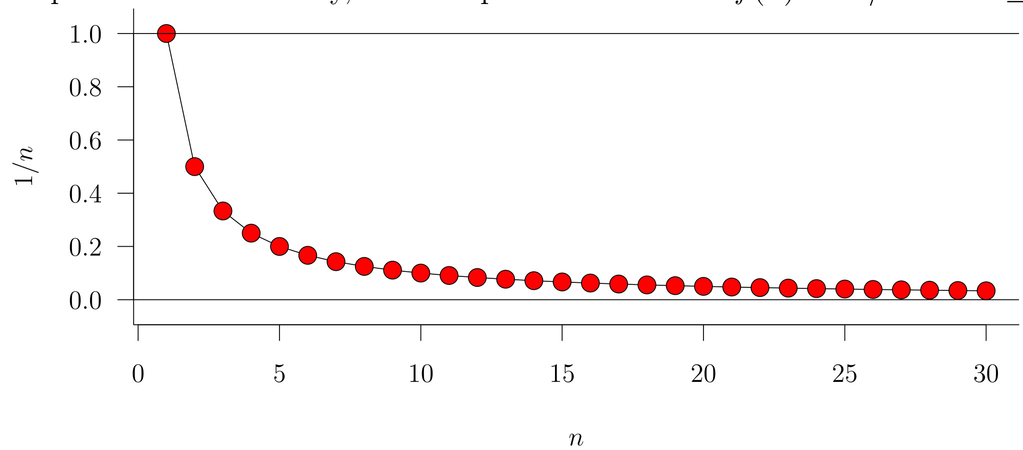
   To justify the entries in this table, consider the following:

   (a) $(-2, -1)$, like any finite open interval, is bounded above and below, with endpoints that are its inf and sup. But the endpoints are not in the set, so neither the min nor max exists.

   (b) Since $n \geq 1$ for all $n \in \mathbb{N}$, we have $1/n \leq 1$ for all $n \in \mathbb{N}$. For $n = 1$ we have $1/n = 1$, so the max (and hence sup) is 1. Since $n > 0$ for all $n \in \mathbb{N}$, we have $1/n > 0$ for all $n \in \mathbb{N}$. But for any $\varepsilon > 0$ there is a natural number n such that $1/n < \varepsilon$. Hence $\inf\{\frac{1}{n} : n \in \mathbb{N}\} = 0$. However, 0 is not in the set, so there is no min. We can visualize this set by plotting the the points with, say, $n \leq 30$ on the number line.

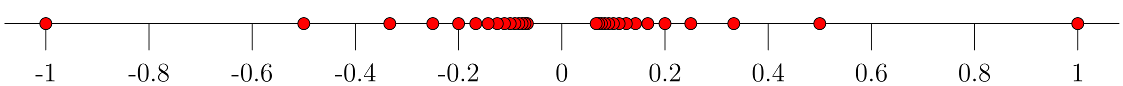


Perhaps more instructively, we can plot the function $f(n) = 1/n$ for $n \leq 30$.

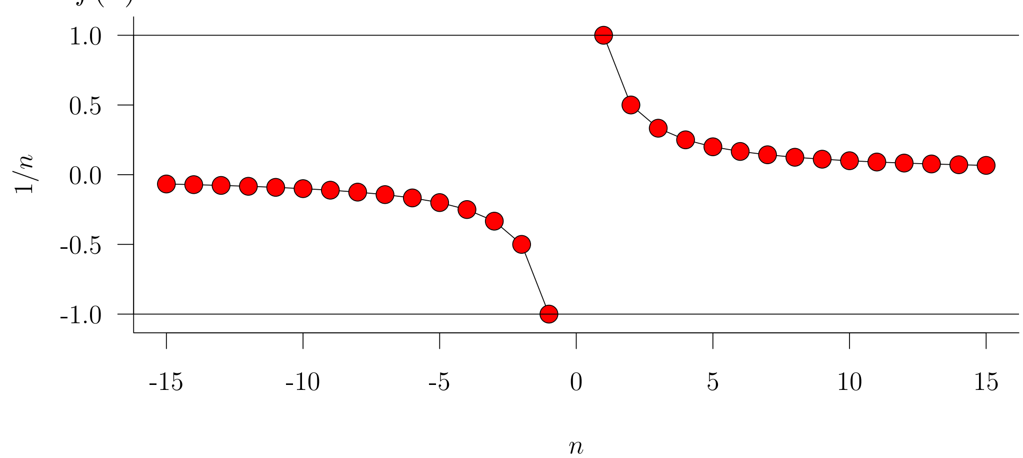


(c) This is similar to (b) except that n can be negative. In particular $n = -1$ is possible, so $1/(-1) = -1$ is in the set and this is the set's minimum.

Number line visualization:

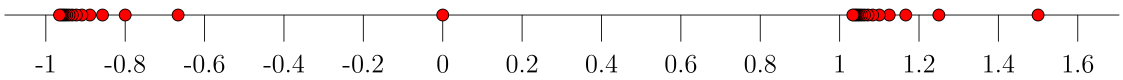


Plot of $f(n)$:

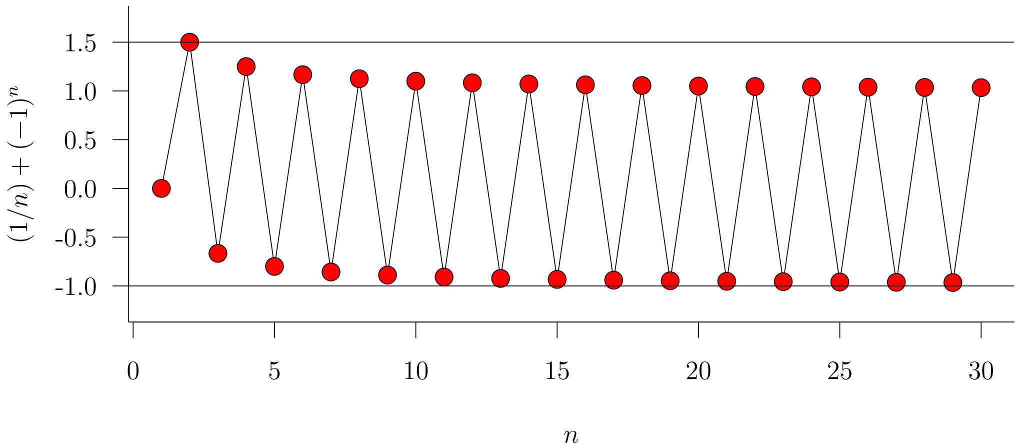


(d) For the last set, let $f(n) = \frac{1}{n} + (-1)^n$ and note that $f(n) \leq 0$ if n is odd and $f(n) > 0$ if n is even. Hence we can restrict attention to odd n to investigate lower bounds and even n to investigate upper bounds. For odd n, we have $f(n) = -1 + \frac{1}{n} > -1$, but $f(n)$ is arbitrarily close to -1 for sufficiently large n. Consequently, the set's inf is -1, which is not in the set, so there is no minimum. For even n, we have $f(n) = 1 + \frac{1}{n} \leq \frac{3}{2}$ for all $n \geq 2$. Since $\frac{3}{2}$ is in the set (for $n = 2$), it is both the sup and max.

Visualizing this set on the number line yields:



while plotting $f(n)$ for $n \leq 30$ yields:



4. Prove that if

$$|x - x_0| < \frac{\varepsilon}{2} \qquad \text{and} \qquad |y - y_0| < \frac{\varepsilon}{2}, \tag{$\heartsuit$}$$

then

$$\begin{aligned} |(x+y) - (x_0+y_0)| &< \varepsilon\,, \\ \text{and } |(x-y) - (x_0-y_0)| &< \varepsilon\,. \end{aligned}$$

Solution: Although it was not stated explicitly, the intention was to assume that $x, y, x_0, y_0, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. Since $\mathbb{R}$ is a field, we have

$$\begin{aligned} |(x+y) - (x_0+y_0)| &= |(x-x_0) + (y-y_0)| && \text{field axioms A1, A2, AM1} \\ &\leq |x-x_0| + |y-y_0| && \text{triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && \text{given inequalities } (\heartsuit) \\ &= \varepsilon\,. \end{aligned}$$

The proof that $|(x-y) - (x_0-y_0)| < \varepsilon$ is similar.

5. Suppose q is a rational number such that

$$1 < q \leq 3 - \sqrt{2}\,. \tag{*}$$

Write $q = m/n$, where $m, n \in \mathbb{N}$ and $\gcd(m,n) = 1$. Prove that $\sqrt{2}$ is a positive distance from q. Specifically, show that

$$\left|\sqrt{2} - \frac{m}{n}\right| \geq \frac{1}{3n^2}\,. \tag{**}$$

Hint: First prove that if $\gcd(m,n) = 1$ then $2n^2$ and m^2 are distinct integers; then "rationalize the numerator" in the LHS of (**).

Solution: Following the hint, let's first prove that if $\gcd(m,n) = 1$ then $2n^2$ and m^2 are distinct integers. If neither m nor n has a factor of 2 then $2n^2$ is even and m^2 is odd, so the two integers are distinct. Suppose that n contains a factor 2. Then m does not contain a factor of 2 (since $\gcd(m,n) = 1$) so, again, $2n^2$ is even and m^2 is odd. Finally, suppose m contains a factor of 2, in which case n does not contain a factor of 2. In this case, m^2 has a factor of 2^2 whereas the largest power of 2 in $2n^2$ is just 1. Hence $2n^2$ and m^2 are distinct. QED.

Now, since $2n^2$ and m^2 are distinct integers, it follows that $|2n^2 - m^2| \geq 1$. Consequently,

$$\begin{aligned}
\left|\sqrt{2} - \frac{m}{n}\right| &= \left|\frac{\sqrt{2}n - m}{n}\right| \\
&= \frac{\left|\sqrt{2}n - m\right|}{n} \cdot \frac{\sqrt{2}n + m}{\sqrt{2}n + m} \\
&= \frac{|2n^2 - m^2|}{n(\sqrt{2}n + m)} \\
&\geq \frac{1}{n(\sqrt{2}n + m)} && \because\ \left|2n^2 - m^2\right| \geq 1 \\
&= \frac{1}{n^2(\sqrt{2} + (m/n))} \\
&\geq \frac{1}{n^2(3)} && \because\ \sqrt{2} + (m/n) \leq 3 \text{ from } (*) \\
&= \frac{1}{3n^2}\,,
\end{aligned}$$

as required. □

Remark: To make what we've proved here a little more concrete, consider the following. Naïvely, we might guess that there is a rational number q in the interval $[1.1, 1.5]$ that is equal to $\sqrt{2}$. Our analysis above restricted attention to rational numbers that satisfy (*), *i.e.,* rational numbers q such that $1 < q \leq 3 - \sqrt{2}$. But $1 < 1.1$ and $1.5 < 3 - \sqrt{2} \simeq 1.5857864$, so the interval $[1.1, 1.5]$ is within the interval to which our analysis applies. Suppose that we were to guess that $\sqrt{2} = 1.4$ (*i.e.,* that $\sqrt{2}$ is exactly equal to 1.4). Noting that $1.4 = 7/5$, our proof implies that, in fact,

$$\left|\sqrt{2} - 1.4\right| \geq \frac{1}{3 \cdot 5^2} = \frac{1}{75} \simeq 0.0133333\,. \tag{1}$$

This is, of course, consistent with a calculator telling us that $\sqrt{2} - 1.4 \simeq 0.0142136$, but no computation with a calculator yields a proof that $\sqrt{2} \neq 1.4$.